Quantitative and Covariational Reasoning in Mathematics

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Introduction

Several scholars have demonstrated the affordances of quantitative *reasoning* (Smith & Thompson, 2007; Thompson, 1990, 2011) and *covariational reasoning* (Carlson et al., 2002; Saldanha & Thompson, 1998; Thompson, 1994b) for learning a variety of ideas in algebra, precalculus, and calculus. The situation presented in Table 1 below provides a context in which to discuss quantitative reasoning.

Table 1

Quantitative Situation

Two brothers, Everett and Harrison, run a 400-meter race against each other. Since Everett is faster, he gives Harrison a 50-meter head start. Everett runs at a constant speed of 6.7 meters per second and Harrison runs at a constant speed of 5.4 meters per second. Both brothers run up to their respective starting lines so that they run the entire race at their respective constant speeds.

Quantitative Reasoning

Quantitative reasoning is a characterization of the mental actions involved in conceptualizing situations in terms of *quantities* and *quantitative relationships*. A *quantity* is an attribute, or quality, of an object that admits a measurement process (Thompson, 1990). One has conceptualized a quantity when she has identified a particular quality of an object and has in mind a process by which she may assign a numerical value to this quality in an appropriate unit (Thompson, 1994b). There are many quantities that one may conceive while thinking about the situation in Table including: *Everett's race distance*, *the time elapsed since Everett passed his starting line*, *Everett's running speed* (all attributes of the race Everett runs), *distance of Harrison's head start*, *the time elapsed since Harrison passed his starting line*, and *Harrison's running speed* (all attributes of the race Harrison runs). It is important to note that quantities do not reside in objects or situations, but are instead constructed in the mind of an individual perceiving and interpreting an object or situation. Quantities are therefore conceptual entities (Thompson, 2011).

Conceptualizing a quantity does not require that one assign a numerical value to a particular attribute of an object. Instead, it is sufficient to simply have a measurement process in mind and to have conceived, either implicitly or explicitly, an appropriate unit. *Quantification* is the process by which one assigns numerical values to some quality of an object (Thompson, 1990). Note that one need not engage in a quantification process in order to have conceived a quantity, but must have in mind a quantification process whereby she may assign numerical values to the quantity (Thompson, 1994b).

The quantities that one may construct upon analyzing a situation are not limited to those whose numerical values are provided, or attainable from direct measurements. For instance, considering the situation presented in Table 1, one may recognize as a quantity *the total amount of time it takes for Everett to complete the race*. As stated above, doing so involves conceptualizing a means of quantification (i.e., imagining a way to assign a numerical value to this attribute of Everett's race). Defining a process by which one may assign numerical values to this quantity involves an operation on two previously defined quantities, *Everett's race distance* and *Everett's running speed*. Since Everett runs at a constant speed for a fixed distance (i.e., the quantities *Everett's race distance* and *Everett's running speed* do not vary), then the total amount of time it takes for Everett to complete the race is given by the ratio of Everett's race distance to his running speed. Furthermore, because conceptualizing a way to quantify *the total amount of time it takes for Everett to complete the race* involves an operation on these two other quantities, we say that this new quantity results from a *quantitative operation*—its conception involved an operation on two other quantities. Quantitative operations result in a conception of a single quantity while also defining the relationship among the quantity produced and the quantities operated upon to produce it (Thompson, 1990, p. 12)[1]. It is for this reason that quantitative operations assist in one's comprehension of a situation (Thompson, 1994b).

It is important to note the distinction between a quantitative operation and a numerical or arithmetic operation. Arithmetic operations are used to calculate a quantity's value whereas quantitative operations define the relationship between a new quantity and the quantities operated upon to conceive it (Thompson, 1990). In the example above, the process by which one may quantify *the total amount of time it takes for Everett to complete the race* involves an operation on the measures of two other quantities (*Everett's race distance* and *Everett's running speed*). Therefore, imagining a way to quantify this new attribute of Everett's race simultaneously defines a relationship between the new quantity and the two quantities operated upon to measure it. Alternatively, claiming that the total amount of time it takes Everett to complete the race is 400/6.7 seconds is a numerical operation, not necessarily a quantitative one. While this numerical operation may proceed from one's construction of *the total amount of time it takes for Everett to complete the race* as a quantitative operation, it is not necessarily the case. Should one not have in mind the quantities whose respective values are 400 and 6.7, then the statement, "The total amount of time it takes Everett to complete the race is 400/6.7 seconds" does not define a relationship between quantities but is rather a statement of fact. Therefore, numerical operations do not guarantee that one has constructed a *quantitative relationship*—"the conception of three quantities, two of which determine the third by a quantitative operation" (Thompson, 1990, p. 13). As Thompson (2011) notes, "Quantitative and numerical operations

[1] The page number cited here, as well as in all subsequent references to Thompson (1990), refers to the non-publication draft available at http://www.patthompson.net/Publications.html.

are certainly related developmentally, but in any particular moment, they are not the same" (p. 42). Other quantitative operations that may be deduced from the task in Table 1 include the following:

- *Harrison's race distance* (determined by the difference of *Everett's race distance* and *the distance of Harrison's head start*),
- *Harrison's total race time* (determined by the ratio of *Harrison's race distance* and *Harrison's running speed*),
- *Everett's distance run after some number of seconds since passing his start line* (determined by the product of *Everett's running speed* and *the number of seconds elapsed since Everett passed his start line*),
- *Harrison's distance run after some number of seconds since passing his start line* (determined by the product of *Harrison's running speed* and *the number of seconds elapsed since Harrison passed his start line*),
- *the distance between Everett and Harrison after any number of seconds since the brothers passed their respective starting lines* (determined by the absolute value of the difference of *Everett's distance run after some number of seconds since passing his start line* and *Harrison's distance run after the same number of seconds since passing his start line*), and
- *the win margin* as a distance (determined by the difference of *Everett's race distance* and *the distance that Harrison has run after the number of seconds it takes for Harrison to finish the race*)[2] or as a time (determined by the absolute value of the difference of *the*

[2] This assumes one is aware of the fact that Everett wins the race.

total amount of time it takes for Everett to complete the race and *the total amount of time it takes for Harrison to complete the race*).

One may clearly deduce several quantitative operations in the simple situation presented in Table 1. Each of these quantitative operations implies a different quantitative relationship. Achieving a complete understanding of the situation involves coordinating these quantitative relationships into a coherent network, or *quantitative structure*. The diagram in Figure 1 illustrates such a structure that one may construct after analyzing the situation in Table . The process of constructing a quantitative structure is called *quantitative reasoning*—the analysis of a situation into a network of quantitative relationships (Thompson, 1990)—and results in achieving a quantitative understanding of the situation. Thompson (1994) explains, "A person comprehends a situation quantitatively by conceiving of it in terms of quantities and quantitative operations. Each quantitative operation creates a relationship: The quantities operated upon with the quantitative operation in relation to the result of operating" (p. 14).

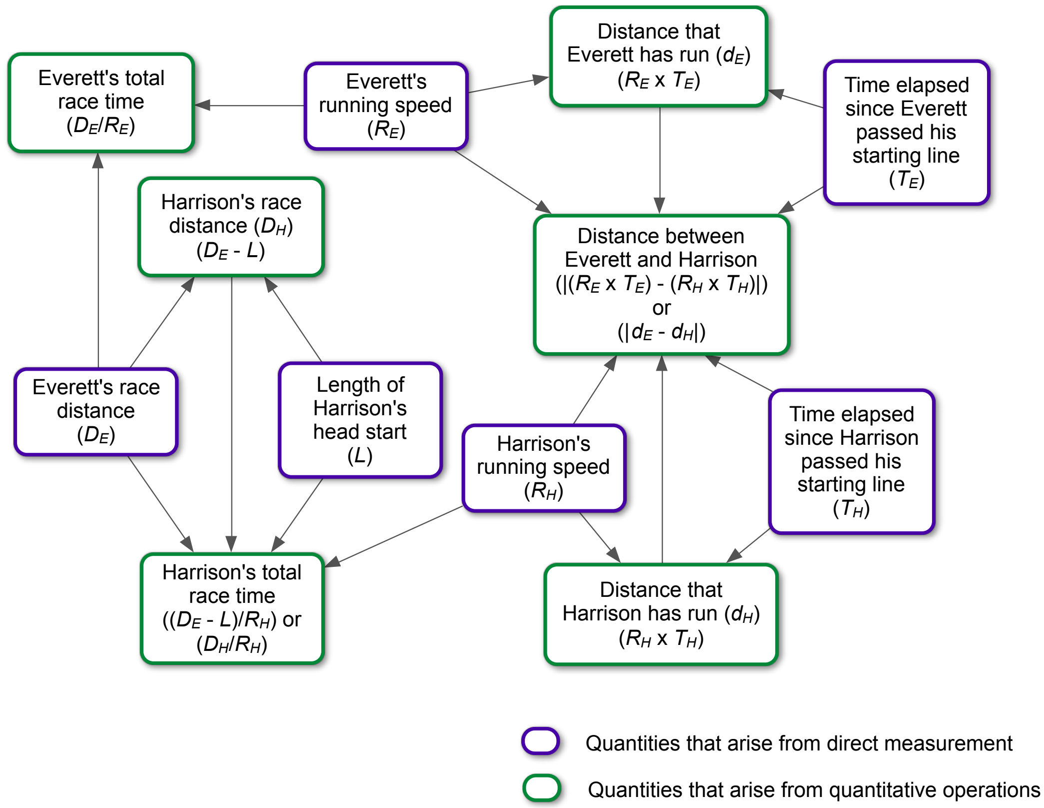

Figure 1. Quantitative structure.

A growing body of research (e.g., Castillo-Garsow, 2010; Confrey & Smith, 1995, Ellis, 2007, Moore, 2012, 2014; Moore & Carlson, 2012; Oehrtman, Carlson, & Thompson, 2008; Thompson 1994b, 2011) has identified quantitative reasoning as a particularly advantageous way of thinking for supporting students' learning of a wide variety of pre- and post-secondary mathematics concepts. Additionally, this body of research has demonstrated the diagnostic and explanatory utility of quantitative reasoning as a theory for how one may conceptualize quantitative situations.

Covariational Reasoning

Covariational reasoning refers to the mental actions involved in coordinating the values of two varying quantities while attending to how these values change in relation to each other (Carlson et al., 2002).

A study by Saldanha and Thompson (1998) gained insight into the mental operations involved in students conceptualizing and reasoning about the continuous covariation of quantities. The researchers conducted a teaching experiment with one 8th grade student to test their hypothesis that students' engagement with tasks requiring the coordination of two sources of information simultaneously is favorable for conceiving of a graph as composed points that record the simultaneous state of two covarying quantities. According to Saldanha and Thompson, covariation entails coupling two quantities so that one may form a *multiplicative object* of the two quantities (p. 1-2). When forming a multiplicative object of two quantities, one develops the immediate and persistent realization that for every possible value that a given quantity can assume, the other quantity also has a value (Saldanha & Thompson, 1998, p. 2).

To Saldanha and Thompson (1998), images of covariation are developmental. An early developmental stage involves one's non-simultaneous coordination of the values of two quantities (i.e., one attends to the value of a quantity, then the value of other, then the value of the first, and so on). In a slightly more sophisticated form of covariational reasoning, one understands time as a continuous quantity, which supports the realization that two quantities' values persist. More sophisticated still is the ability to imagine both quantities being tracked for some duration and recognize the correspondence between the two quantities as an emergent property of the image. Saldanha and Thompson describe *continuous covariation* as the understanding that if a quantity assumes different values at different moments in time, the quantity assumed all intermediary values during this interval of time.

Carlson, et al. (2002) propose a framework for characterizing students' mental actions while engaged in tasks involving dynamic function events. Their theoretical framework consists of a hierarchy of five mental actions of covariational reasoning along with five corresponding covariational reasoning levels. The authors define the first mental action (MA 1, *coordination of quantities*) as an individual's recognition that a change in the value of one quantity corresponds to a change in the value of another. The second mental action (MA 2, *coordination of direction of change*) describes not only a recognition that the values of two quantities vary in tandem, but requires one to coordinate the direction of change in the value of one quantity with changes in the value of another. The third mental action (MA 3, *coordination of amounts of change*) involves one in attending to the amount of change in the value of one quantity with respect to the amount of change in the value of another. The fourth mental action (MA 4, *average rate of change*) focuses on one's capacity to attend to the average rate of change of the value of one quantity with respect to the value of another. Finally, the fifth mental action (MA 5, *instantaneous rate of change*) describes one's ability to attend to the instantaneous rate of change of the value of one quantity with respect to the value of another. Carlson et al. explain that the purpose of their proposed framework is to aid in the evaluation of covariational thinking to a greater extent than had been done previously.

Thompson (2011) provides an additional account of the mental operations involved in conceptualizing and coordinating the simultaneous variation of the values of two quantities. He explains that to imagine variation in a quantity's value is to expect the value of the quantity to differ at two different moments in (conceptual) time and to realize that the quantity's measure assumed all values between the measure of the quantity at the beginning of the interval of time and the measure of the quantity at the end of the interval of time. Thinking about continuous variation therefore amounts to first anticipating an interval of time over which variation in a quantity's value occurs, which allows one to expect that the quantity's value will vary by a specific amount over this interval of time. One then imagines the quantity's value varying in microscopic bits, each of which occurs over a very small interval of time. To imagine a quantity's value varying continuously, one must realize that variation occurred within these very small intervals of time (i.e., one has to imagine that for every value between the initial and final values that the quantity assumed over this small interval of variation, there was a time within the small interval of time over which the variation occurred that the quantity assumed this value).