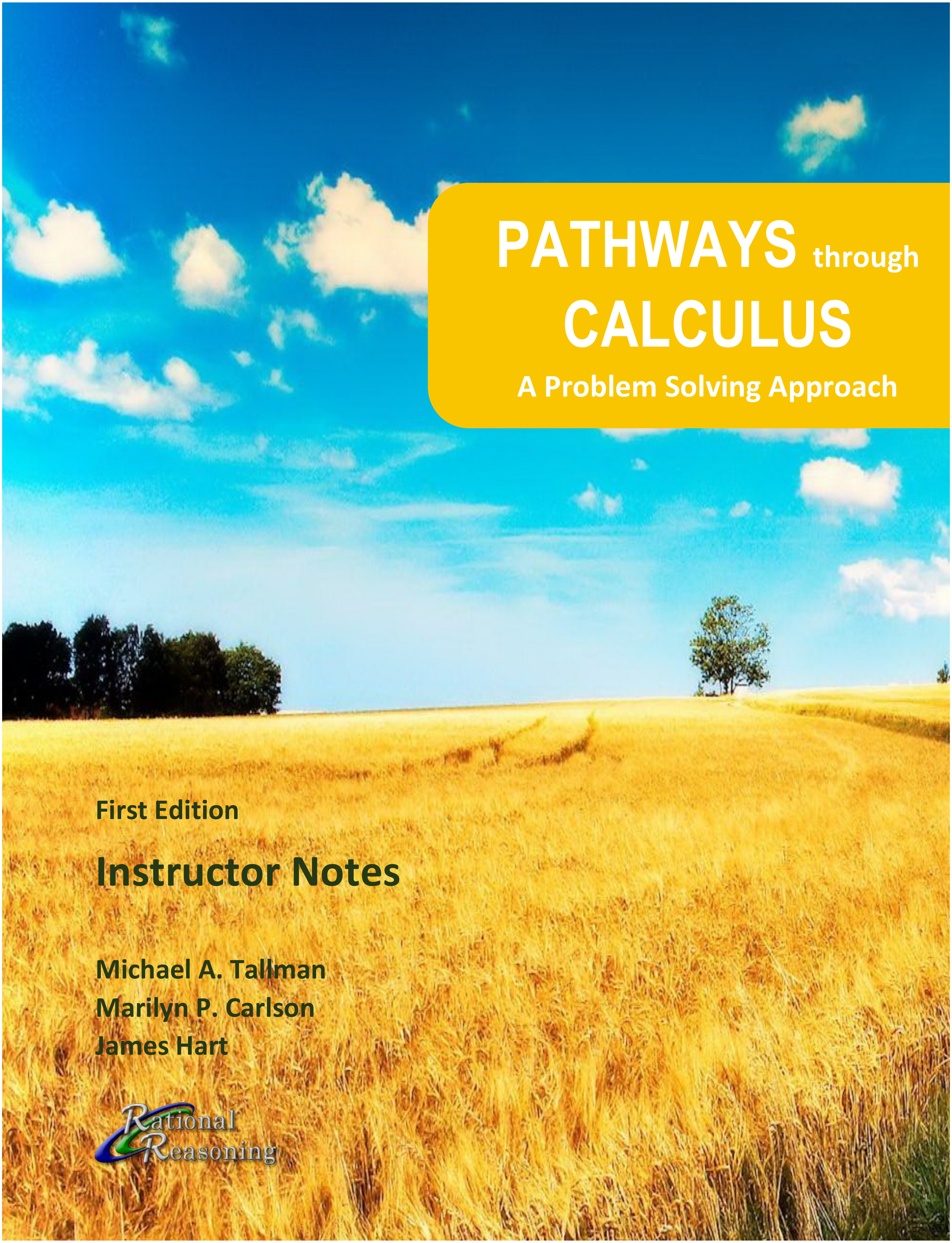

# PATHWAYS through CALCULUS

## A Problem Solving Approach

First Edition

## Instructor Notes


Michael A. Tallman
Marilyn P. Carlson
James Hart


Rational Reasoning

# Pathways Through Calculus

## Instructor Notes

*First Edition*

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# *Introduction*

# *Overview of Workbook Content*

This workbook contains investigations that we designed to help you build your own understanding of the central ideas of calculus. Many, but not all, of the investigations are accompanied by homework problems. We have intentionally included homework problems in that are much like problems in the investigations. This is so you can develop the thinking and understandings that will be needed to be successful in calculus and future mathematics, science, and engineering courses.

The first two investigations review average and constant rate of change as well as linearity. These are essential prerequisite concepts for calculus. Investigation 3 explores the idea of local linearity, an important concept that underlies the notion of a derivative function, the main idea of Calculus I. Investigations 4–6 carefully develop your understanding of derivative functions by leveraging the meanings emphasized in Investigations 1–3. Investigation 7 introduces a method for computing the derivative of composite functions, again by leveraging the understandings developed in Investigations 1 and 2. Investigations 8 and 9 require you to apply your understanding of derivatives to solve novel problems. Finally, Investigation 10 introduces the notion of an accumulation function.

By engaging with the questions in the investigations and homework, your reasoning and problem-solving abilities will get better and better. Over time you will become a powerful mathematical thinker who has confidence in your ability to solve novel problems on your own.

To: The Calculus Student

# *Welcome!*

You are about to begin a new mathematical journey that we hope will lead to your choosing to continue to study mathematics. Even if you do not currently view yourself as being particularly talented at mathematics, it is very likely that these materials and this course will change your perspective. The materials in this workbook were designed with student learning and success in mind and are based on decades of research on mathematics teaching and learning. In addition to becoming more confident in your mathematical abilities, the reasoning patterns, problem solving abilities, and content knowledge you acquire will make more advanced courses in mathematics, the sciences, engineering, nursing, and business more accessible. The investigations will help you see a purpose for learning and understanding the ideas of calculus, while also helping you acquire critical knowledge and ways of thinking that will support your learning in future mathematics, science, and engineering courses. To assure your success, we ask that you make a strong effort to make sense of the questions you encounter. This will assure that your mathematical journey through this course is rewarding and transformational.

Wishing you much success!

Dr. Michael A. Tallman, Dr. Marilyn P. Carlson, Dr. James Hart

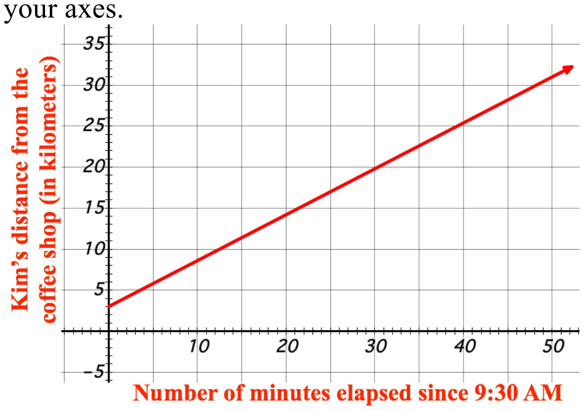
# *Table of Contents*

The focus of this investigation is on building students' intuitive understanding of constant rate of change. By the end of the investigation we want students to understand that if the measure of one quantity varies at a constant rate with respect to the measure of another, then the changes in the measures of the quantities are proportional. We leverage this meaning of constant rate of change to support students' construction of the point-slope form of a linear equation. In particular, this investigation supports students' understanding that if a constant rate of change of $y$ with respect to $x$ exists, then $\Delta y = m \cdot \Delta x$ for some constant $m$. Students will use this fact to determine changes in one variable given the changes in another variable and will find the constant rate of change $m$ given corresponding changes in two variables or two ordered pairs. We believe that focusing most of our attention on the formula $\Delta y = m \cdot \Delta x$ instead of $m = {}^{\Delta y}/_{\Delta x}$ is a better foundation for building a meaningful understanding of linear function formulas, which are based on the idea of starting with a given reference point, calculating a change in $x$ away from this reference point, and using the constant rate to determine the corresponding change in $y$ away from the reference point, which is then used to determine the new value of $y$. We also believe it helps students understand the constant rate of change as a tool to find function values for any change in the input value and not just integer changes. For example, it helps students understand that if $m = {}^{3}/_{5}$ then for *any* change in the value of $x$, the change in the value of $y$ will be ${}^{3}/_{5}$ times as large, as opposed to students thinking in "rise over run" terms, which usually translates into students thinking that they must change the value of $x$ in intervals of 5 and the value of $y$ in intervals of 3. Also, "rise over run" tends to suggest that students should vary $y$ before they vary $x$, which typically hampers their understanding of how functions are defined and the role of the independent and dependent quantities in co-variation.

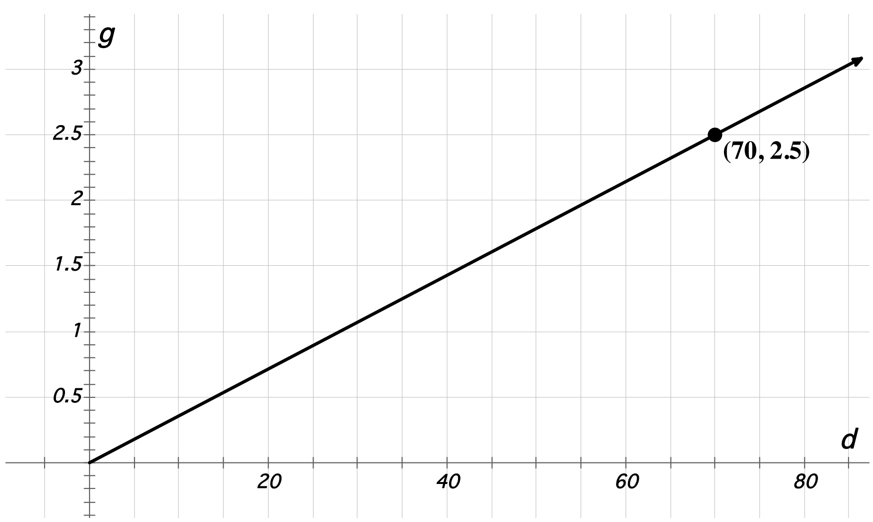
1. Suppose Kim is riding her bike along a straight road at a constant rate of 0.56 km/min. Kim passes a coffee shop while traveling at this constant rate. At 9:30 AM, Kim is 3 km past the coffee shop.
   a. How far is Kim from the coffee shop at 9:31 AM?
   Since Kim travels at a constant rate of 0.56 km/min, we know that she traveled 0.56 km in the one minute between 9:30 AM and 9:31 AM. Since Kim was 3 km past the coffee shop at 9:30 AM, she is 3.56 km past the coffee shop at 9:31 AM.
   b. How far is Kim from the coffee shop at 10:17 AM?
   Since Kim travels at a constant rate of 0.56 km/min, we know that she traveled 0.56(47) km, or 26.32 km, in the 47 minutes between 9:30 AM and 10:17 AM. Since Kim was 3 km past the coffee shop at 9:30 AM, she is 29.32 km past the coffee shop at 10:17 AM.
   c. How far is Kim from the coffee shop 24.6 minutes past 9:30 AM?
   Since Kim travels at a constant rate of 0.56 km/min, we know that she traveled 0.56(24.6) km, or 13.776 km, in the 24.6 minutes after 9:30 AM. Since Kim was 3 km past the coffee shop at 9:30 AM, she is 16.776 km past the coffee shop 24.6 minutes past 9:30 AM.
   d. Define a function that relates Kim's distance from the coffee shop (in kilometers) in terms of the number of minutes elapsed since Kim was 3 km past the coffee shop. Be sure to define your variables.
   Let $d$ represent Kim's distance from the coffee shop in kilometers and let $t$ represent the number of minutes elapsed since Kim was 3 km past the coffee shop (i.e., since 9:30 AM). Then $d = 0.56t + 3$.
   e. Let $\Delta t$ represent a change in the number of minutes elapsed while Kim is riding her bike at a constant rate and let $\Delta d$ represent the corresponding change in the number of kilometers Kim traveled. How are $\Delta t$ and $\Delta d$ related? Write an equation that expresses the relationship between $\Delta t$ and $\Delta d$.
   Since Kim is riding her bike at a constant rate of 0.56 km/min, $\Delta d$ is always 0.56 times as large as $\Delta t$. Therefore, $\Delta d = 0.56\Delta t$.
   f. Use your response to Part (e) to determine what time Kim passed the coffee shop.

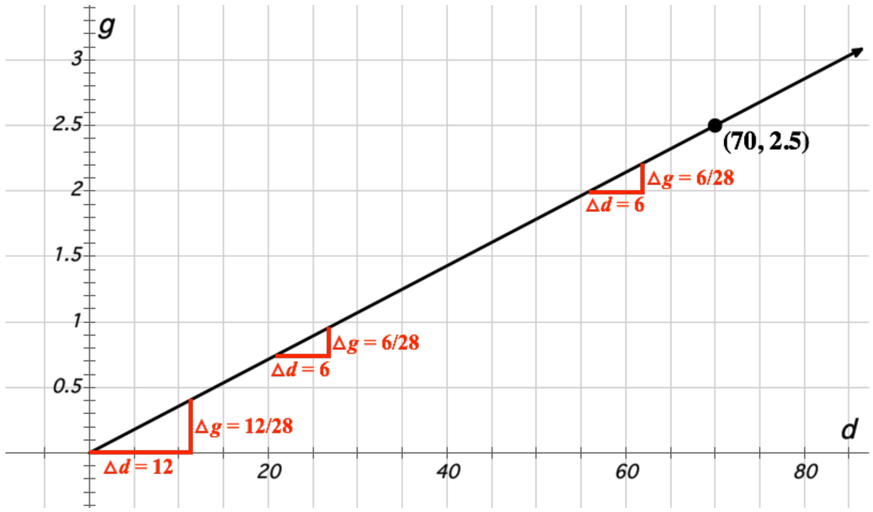
At 9:30 AM Kim is 3 km past the coffee shop. We are interested in determining the amount of time it took Kim to ride 3 km beyond the coffee shop; that is, we are interested in determining the change in time that corresponds to a change in distance of 3 km. Since Kim is riding her bike at a constant rate of 0.56 km/min, $\Delta d = 0.56\Delta t$. Since $\Delta d = 3$, we have $\Delta t = {}^{3}/_{0.56} = 5.357$. This change in time is approximately 5 minutes and 21 seconds. Therefore, Kim passed the coffee shop at approximately 9:24:39 AM.

g. Sketch a graph of the relationship between Kim's distance from the coffee shop (in kilometers) and the number of minutes elapsed since Kim was 3 km past the coffee shop. Be sure to label your axes.




2. John Paul is driving on Interstate 35 from Norman, OK to Stillwater, OK. John Paul's car consumes fuel at a constant rate while he drives on I-35. The graph below represents the relationship between the number of miles John Paul has driven on I-35 (represented by the variable $d$) and the number of gallons of fuel his car has consumed since he started driving on I-35 (represented by the variable $g$). The point (70, 2.5) is on the graph, as indicated.






a. As the number of miles that John Paul has driven on I-35 changes from 0 to 12 miles, how much does the amount of fuel his car has consumed change? Represent this change on the graph and explain how you determined this change.
Since John Paul's car consumed 2.5 gallons of fuel after he drove 70 miles on I-35 (as indicated by the graph), we conclude that John Paul's car consumes fuel at a constant rate of 28 miles per gallon. This means that for any change in the number of gallons John Paul's car has consumed since he started driving on I-35, the change in the number of miles he has driven is 28 times as large (i.e., $\Delta d = 28\Delta g$). We can also think about this constant rate of change in the following way: For any change in the number of miles John Paul has driven on I-35, the change in the number of gallons of fuel his car has consumed is 1/28 times as large (i.e., (i.e., $\Delta g = (^1/_{28})\Delta d$). As the number of miles that John Paul has driven on I-35 changes from 0 to 12 miles, the amount of fuel his car consumed changes by $(^1/_{28})12$ gallons.

b. As the number of miles that John Paul has driven on I-35 changes from 21 to 27 miles, how much does the amount of fuel his car has consumed change? Represent this change on the graph and explain how you determined this change.
We determined in our response to Part (a) that the number of gallons John Paul's car has consumed since he started driving on I-35 changes at a constant rate of 1/28 with respect to the number of miles he has driven on I-35. Therefore, as the number of miles that John Paul has driven on I-35 changes from 21 to 27 miles, the amount of fuel his car has consumed changes by $^1/_{28}(27 - 21)$ gallons, or $^6/_{28}$ gallons.

c. As the number of miles that John Paul has driven on I-35 changes from 56 to 62 miles, how much does the amount of fuel his car has consumed change? Represent this change on the graph and explain how you determined this change.
We determined in our response to Part (a) that the number of gallons John Paul's car has consumed since he started driving on I-35 changes at a constant rate of 1/28 with respect to the number of miles he has driven on I-35. Therefore, as the number of miles that John Paul has driven on I-35 changes from 56 to 62 miles, the amount of fuel his car has consumed changes by $^1/_{28}(62 - 56)$ gallons, or $^6/_{28}$ gallons.

d. How much does the amount of fuel John Paul's car consumed change for any change of 6 miles he has driven on I-35?
Since the number of gallons John Paul's car has consumed since he started driving on I-35 changes at a constant rate of 1/28 with respect to the number of miles he has driven on I-35, John Paul's car consumes 6/28 gallons of fuel for any change of 6 miles he has driven on I-35.

e. As the number of miles that John Paul has driven on I-35 changes from $d_1$ to $d_2$ miles, how much does the amount of fuel his car has consumed change? Explain how you determined this change.
As the number of miles that John Paul has driven on I-35 changes from $d_1$ to $d_2$ miles, the amount of fuel his car has consumed changes by ${}^{1}/_{28}(d_2 - d_1)$ gallons. Since the number of gallons John Paul's car has consumed changes at a constant rate of 1/28 with respect to the number of miles he has driven on I-35, the change in the number of gallons of fuel his car has consumed is 1/28 times as large as the change in the number of miles he has driven on I-35.

f. Define a function that relates the number of gallons of fuel John Paul's car has consumed since he started driving on I-35 in terms of the number of miles he has driven on I-35. Be sure to define your variables.
Let $d$ represent the number of miles John Paul has driven on I-35 and let $g$ represent the number of gallons of fuel his car has consumed since he started driving on I-35. Then $g = {}^{1}/_{28} \cdot d$.

g. Let $\Delta d$ represent a change in the number of miles John Paul has driven on I-35 and let $\Delta g$ represent the corresponding change in the number of gallons of fuel John Paul's car has consumed. How are $\Delta d$ and $\Delta g$ related? Write an equation that expresses the relationship between $\Delta d$ and $\Delta g$.
The change in the number of gallons John Paul's car has consumed since he started driving on I-35 is $1/28^{th}$ times as large as the number of miles he has driven. This relationship is expressed symbolically as $\Delta g = {}^{1}/_{28} \cdot \Delta d$.

h. Determine whether the following two statements are true of false and justify your answer.

  i. T or F: If the *changes* in the values of two quantities are proportionally related, then the *values* of the two quantities are proportionally related.
  This statement if false. In Problem 1 the change in Kim's distance from the coffee shop $\Delta d$ is always 0.56 times as large as the change in time $\Delta t$ since she was 3 km past the foccee shop ($\Delta d = 0.56\Delta t$). However, substituting values into $d$ and $t$ verifies that $d \neq 0.56t$.

  ii. T or F: If the *values* of two quantities are proportionally related, then the *changes* in the values of the quantities are also proportionally related.
  This statement is true. If two quantities $x$ and $y$ are proportionally related, then $y = m \cdot x$. We can also say that as the two quantities change together, they are related by a constant multiple. It is also true that $\Delta y = m \cdot \Delta x$ since any pair of values in the relationship $(x, y)$ can be thought of as changes away from zero.

3. The situations in Questions 1 and 2 involved quantities that varied at a constant rate with respect to each other. Reflect on your response to 1(e) and 2(g) and explain what it means for two quantities to vary at a constant rate. Your explanation should apply to the situations in *both* Problems 1 and 2.
Two quantities change at a constant rate with respect to each other if changes in one quantity are proportional to corresponding changes in the other. Let $x$ and $y$ represent the measures of Quantity A and Quantity B respectively. If Quantity B varies at a constant rate with respect to Quantity A, then $\Delta y = m \cdot \Delta x$ for some constant $m$. The value of $m$ is called the constant rate of change Quantity B with respect to Quantity A.

4. Suppose Quantity A varies at a constant rate of –3.1 with respect to Quantity B. Let $y$ represent the measure of Quantity A and let $x$ represent the measure of Quantity B. When $x$ has a value of 1.7, $y$ has a value of 2.4. Use the meaning of constant rate of change you described in response to Question 3 to answer the following.

   a. Determine the change in the value of $y$ as $x$ changes by 1.9.
   $\Delta x = 1.9$
   $\Delta y = -3.1\Delta x$
   $\Delta y = -3.1(1.9) = -5.89$

b. Determine the change in the value of $x$ as $y$ changes by –5.2.
$\Delta y = -5.2$
$-5.2 = -3.1\Delta x$
$\Delta x = (-5.2)/(-3.1) \approx 1.68$

c. Determine the value of $y$ when $x = 0$.
$\Delta x = 0 - 1.7 = -1.7$
$\Delta y = -3.1\Delta x$
$\Delta y = -3.1(-1.7) = 5.27$
$y = 2.4 + \Delta y = 2.4 + 5.27 = 7.67$

d. Determine the value of $x$ when $y = 4.8$.
$\Delta y = 4.8 - 2.4 = 2.4$
$\Delta y = -3.1\Delta x$
$2.4 = -3.1\Delta x$
$\Delta x = {}^{2.4}/_{-3.1}$
$x = 1.7 + \Delta x = 1.7 + {}^{2.4}/_{-3.1} \approx 0.93$

5. Suppose Quantity A varies at a constant rate of $m$ with respect to Quantity B. Let $y$ represent the measure of Quantity A and let $x$ represent the measure of Quantity B. When $x$ has a value of $x_1$, $y$ has a value of $y_1$. Use the meaning of constant rate of change you described in response to Question 3 to answer the following.
   a. Determine the change in the value of $y$ as $x$ changes by $\Delta x$.
   Since $y$ changes at a constant rate of $m$ with respect to $x$, $\Delta y = m\Delta x$.
   b. Determine the value of $y$ when $x = 0$.
   $\Delta x = 0 - x_1 = -x_1$
   $\Delta y = m\Delta x = m(-x_1)$
   $y = y_1 + \Delta y = y_1 + m(-x_1)$
   c. Determine the value of $y$ when $x = 7.4$.
   $\Delta x = 7.4 - x_1$
   $\Delta y = m\Delta x = m(7.4 - x_1)$
   $y = y_1 + \Delta y = y_1 + m(7.4 - x_1)$
   d. Write an equation that determines the value of $y$ for any value of $x$.
   $\Delta x = x - x_1$
   $\Delta y = m\Delta x = m(x - x_1)$
   $y = y_1 + \Delta y = y_1 + m(x - x_1)$
   $y = y_1 + m(x - x_1)$

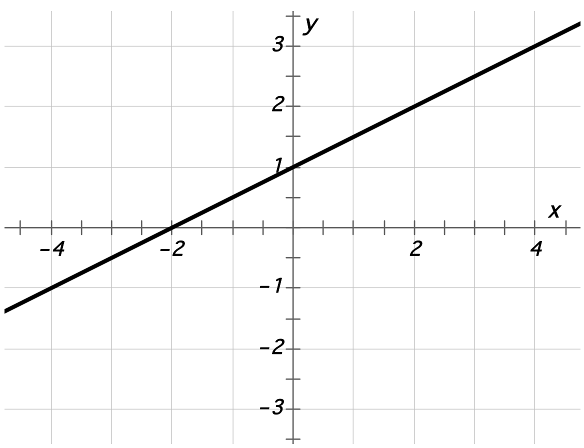
Two quantities ***change at a constant rate*** with respect to each other if changes in one quantity are proportional to corresponding changes in the other.

For example, suppose Quantity A changes at a constant rate with respect to Quantity B. If $y$ represents the measure of Quantity A and $x$ represents the measure of Quantity B, then the change in $y$ is proportionally related to the change in $x$. This means that for any change in $x$, the corresponding change in $y$ is always the same number of times as large. If we let $m$ denote the number of times larger $\Delta y$ is than $\Delta x$, we can express the proportional relationship between changes in the measures of Quantity A and Quantity B as $\Delta y = m\Delta x$. The value of $m$ is called the ***constant rate of change*** of Quantity A with respect to Quantity B.

6. Give three examples of pairs of quantities that vary at a constant rate with respect to each other. Explain why each pair of quantities are related by a constant rate of change.

*Ex. 1.* Sliced ham at the local grocery store costs $1.19 per pound. The total cost of an order of ham varies at a constant rate with respect to the number of pounds of ham purchased.
*Ex.* 2. As a candle burns, the remaining length of the candle varies at a constant rate with respect to the amount of time it has been burning.
*Ex. 3.* While driving on the freeway with the cruise control on, the distance driven since the cruise control was turned on varies at a constant rate with respect to the time elapsed since the cruise control was turned on.

7. Suppose $x$ and $y$ represent the measures of two quantities that vary at a constant rate with respect to each other. For Parts (a) – (c) below, use the given information to write a formula that defines the relationship between $x$ and $y$.
   a. $y$ changes at a constant rate of –0.9 with respect to $x$.
   $y = 2.4$ when $x = -5.8$.
   $\Delta y = -0.9\Delta x$
   $\Delta y = -0.9(x - (-5.8)) = -0.9(x + 5.8)$
   $y = 2.4 + \Delta y$
   $y = 2.4 - 0.9(x + 5.8)$
   b. $y = 3.6$ when $x = 12.2$.
   $y = -1.5$ when $x = 8.7$.
   $\Delta y = m\Delta x$
   $m = {}^{\Delta y}/_{\Delta x} = {}^{(-1.5 - 3.6)}/_{(8.7 - 12.2)} = {}^{5.1}/_{-3.5}$
   $y = 3.6 + m(x - 12.2)$ or $y = -1.5 + m(x - 8.7)$
   $y = 3.6 + ({}^{5.1}/_{-3.5})(x - 12.2)$ or $y = -1.5 + ({}^{5.1}/_{-3.5})(x - 8.7)$
   c.




   We notice that the points (0, 1) and (4, 3) are on the graph of the function.
   $\Delta y = m\Delta x$
   $m = {}^{\Delta y}/_{\Delta x} = {}^{(3 - 1)}/_{(4 - 0)} = ½$
   $y = 1 + m(x - 0)$ or $y = 3 + m(x - 4)$
   $y = 1 + ½(x)$ or $y = 3 + ½(x - 4)$

8. Suppose $x$ and $y$ represent the measures of two quantities that vary at a constant rate with respect to each other. For Parts (a) and (b) below, use the given information to write a formula that defines the relationship between $x$ and $y$.
   a. $y$ changes at a constant rate of $m$ with respect to $x$.
   $y = y_1$ when $x = x_1$.
   $\Delta y = m\Delta x$

$\Delta x = x - x_1$
$y = y_1 + \Delta y = y_1 + m\Delta x$
$y = y_1 + m(x - x_1)$

b. $y = y_1$ when $x = x_1$.
$y = y_2$ when $x = x_2$.
$\Delta y = m\Delta x$
$m = {}^{\Delta y}/_{\Delta x} = (y_2 - y_1)/(x_2 - x_1)$
$y = y_1 + ((y_2 - y_1)/(x_2 - x_1))(x - x_1)$ or $y = y_2 + ((y_2 - y_1)/(x_2 - x_1))(x - x_2)$

**Homework**

1. What does it mean for an object to move at a constant speed? (*Note: Please say something more than "The speed doesn't change." Be descriptive and reference specific quantities.*)
   Explanations will vary. Some important observations could include the following.
   For an object traveling at a constant speed, the same change in time elapsed will always correspond to equal changes in distance traveled, and likewise the same change in distance traveled will correspond to equal changes in time elapsed. Furthermore, if we know how far the object travels in some total amount of time, then in some fraction of that time it will travel the same fraction of the total distance.

2. Paul was walking in a park. Assume that he walked at a constant speed during the entire trip, and also suppose that during one part of the trip he walked 52.8 feet in 8 seconds.
   a. Provide at least four conclusions we can draw from the given information.
   Answers will vary. Some examples are given. Paul walked 26.4 feet (½ times as far as 52.8 feet) in ½ times as long as 8 seconds (or 4 seconds). He walked 79.2 feet (3 times as far as 52.8 feet) in 24 seconds (3 times as long as 8 seconds). He walked $\frac{1}{8}$ times as far as 52.8 feet (or $\frac{1}{8}(52.8) = 6.6$ feet) in 1 second.
   b. How far did Paul walk in 14 seconds?
   We can use the unit rate (in this case 6.6 feet in 1 second) to determine that in 14 seconds Paul walked 14(6.6) or 92.4 feet.
   c. Does your answer to Part (b) depend on which 14-second interval we're talking about? Explain.
   The answer to Part (b) does not depend on a specific 14-second interval. Over *any* 14-second interval while Paul is walking he travels 92.4 feet.
   d. How long did it take Paul to travel any 20-foot distance during his walk?
   It takes Paul $\frac{8}{52.8}$ of a second to travel 1 foot, so to travel 20 feet it takes $20\left(\frac{8}{52.8}\right) = \frac{100}{33} \approx 3.03$ seconds.

3. Suppose we have a partially filled pitcher of water and that we want to add more water to the pitcher. We know that adding 60 ounces of water to the pitcher will increase the height of water in the pitcher by 7.8 inches, and that these two quantities are related by a constant rate of change. Define variables to represent the quantities in this context and then represent the relationship between corresponding changes in these quantities.
   Let $h$ represent the height of the water in the pitcher (in inches) and let $v$ represent the volume of water in the pitcher (in ounces). Then $\Delta h = \frac{7.8}{60} \cdot \Delta v$ or $\Delta h = 0.13 \cdot \Delta v$.

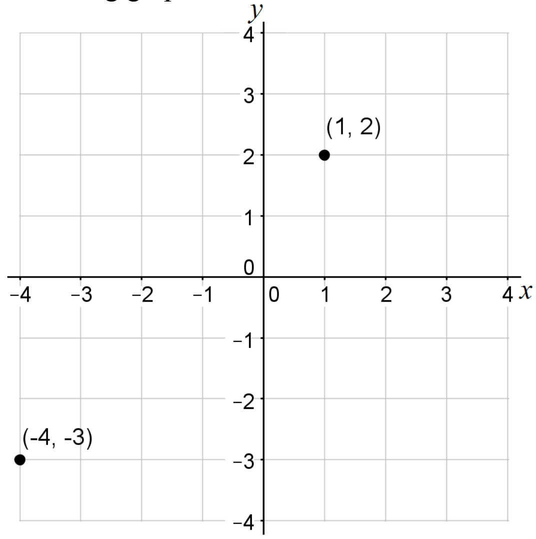
4. Suppose we know that $\Delta y = m \cdot \Delta x$ for some constant $m$, and we are given the information in the following table. What is the value of $m$?

| $x$ | $y$ |
|---|---|
| –3 | 15.5 |
| 1 | 5.5 |
| 3 | 0.5 |
| 8 | –12 |

   We just need to take two entries in the table and calculate $\Delta x$ and $\Delta y$. For example, suppose we select (1, 5.5) and (3, 0.5). Then $\Delta x = 3 - 1 = 2$ and $\Delta y = 0.5 - 5.5 = -5$, so

$$\Delta y = m \cdot \Delta x \qquad \rightarrow \qquad m = \frac{\Delta y}{\Delta x} = \frac{-5}{2} = -2.5.$$

5. Suppose we know that $\Delta y = m \cdot \Delta x$ for some constant $m$, and we are given the information in the following graph. What is the value of $m$?




We use the points on the graph to determine $\Delta x$ and $\Delta y$. So $\Delta x = 1 - (-4) = 5$ and $\Delta y = 2 - (-3) = 5$, so

$$\Delta y = m \cdot \Delta x \qquad \to \qquad m = \frac{\Delta y}{\Delta x} = \frac{5}{5} = 1.$$

6. Suppose we know that the changes in the values of two variables are related according to $\Delta y = 3 \cdot \Delta x$.
   a. If we start off at $x = 5$ and let $x$ change to be $x = 12$,
      i. What is the change in $x$?
         $\Delta x = 12 - 5 = \boxed{7}$
      ii. By how much does $y$ change for the change in $x$ you found in Part (i)?
         $\Delta y = 3 \cdot \Delta x = 3 \cdot (7) = \boxed{21}$
      iii. Suppose we know that $y = -2$ when $x = 5$. What is the value of $y$ when $x = 12$? How did you find this?
         $y = -2 + 21 = \boxed{19}$. We find this by adding the change in $y$ to the starting value of $y$.
   b. If we start off at $x = 7$ and let $x$ change to be $x = -3$,
      i. What is the change in $x$?
         $\Delta x = -3 - 7 = \boxed{-10}$
      ii. By how much does $y$ change for the change in $x$ you found in Part (i)?
         $\Delta y = 3 \cdot \Delta x = 3 \cdot (-10) = \boxed{-30}$
      iii. Suppose we know that $y = 8$ when $x = 7$. What is the value of $y$ when $x = -3$? How did you find this?
         $y = 8 + (-30) = \boxed{-22}$. We find this by adding the change in $y$ to the starting value of $y$.

7. Suppose you have a cell phone plan whose cost is based on the number of minutes you talk. Let $n$ represent the number of minutes talked in a month and let $c$ represent the monthly cost of using your phone (in dollars). Furthermore, suppose $c = 45.70$ when $n = 95$ and that $\Delta c = 0.06 \cdot \Delta n$.
   a. What is the value of $c$ when $n = 325$? What does this tell us?
      $\Delta n = 230$, so $\Delta c = 0.06(230) = 13.80$, and thus $c = 45.70 + 13.80 = 59.50$ when $n = 325$. This tells us that the cost to talk for 325 minutes is \$59.50.

b. What is the value of $c$ when $n = 0$? What does this tell us?
$\Delta n = -95$, so $\Delta c = 0.06(-95) = -5.70$, and thus $c = -5.70 + 45.70 = 40$ when $n = 0$. This tells us that the base cost for your cell phone plan (prior to making any calls and adding the corresponding additional costs) is \$40.

8. Suppose we are given that $\Delta y = 4.5 \cdot \Delta x$ and that when $x = 1$, $y = 4$. We want to know the new value of $y$ when $x = -4$ . Answer the questions that follow.

| | |
|---|---|
| $y = 4.5(x - 1) + 4$ | |
| $y = 4.5(-4 - 1) + 4$ | a. What does $-4 - 1$ represent? |
| $y = 4.5(-5) + 4$ | b. What does $4.5(-5)$ represent? |
| $y = -22.5 + 4$ | c. What does $-22.5 + 4$ represent? |
| $y = -18.5$ | d. What does $-18.5$ represent? |

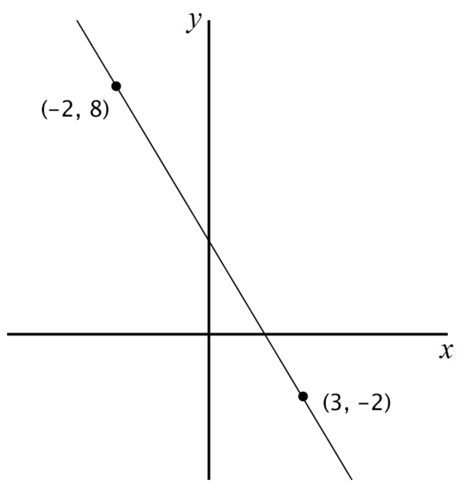
a. $-4 - 1$ represents the change in $x$ from $x = 1$ to $x = -4$.
b. $4.5(-5)$ represents the change in $y$ from $y = 4$ to the new value of $y$.
c. $-22.5 + 4$ represents the value of $y$ when $x = -4$ (because it shows the change in $y$ added to the "initial" value of $y$).
d. $-18.5$ represents the value of $y$ when $x = -4$.

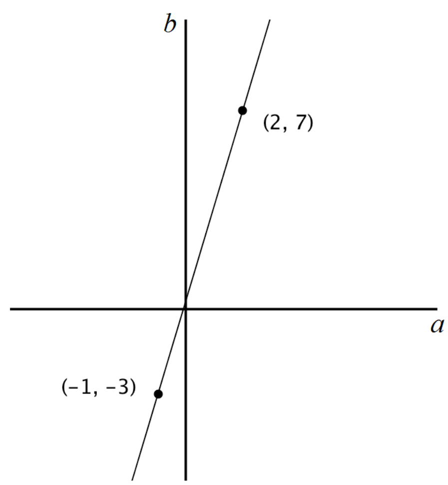
9. The constant rate of change of $y$ with respect to $x$ is 4, and (5, 4) is a point on the graph.
   a. Write the formula for the linear function. $y = 4(x - 5) + 4$
   b. Find the value of $y$ when $x = 2$. $y = 4(2 - 5) + 4 = -8$

10. The constant rate of change of $y$ with respect to $x$ is $-3.2$, and $(-3, -2)$ is a point on the graph.
   a. Write the formula for the linear function. $y = -3.2(x + 3) - 2$
   b. Find the value of $y$ when $x = 5$. $y = -3.2(5 + 3) - 2 = -27.6$

11. Write the formula for each of the linear functions described below.
   a. $y$ changes at a constant rate of 4.8 with respect to $x$, and (7, 9.3) is a point on the graph.
   $y = 4.8(x - 7) + 9$.
   b. $y$ changes at a constant rate of $-1.9$ with respect to $x$, and (4, 6) is a point on the graph.
   $y = -1.9(x - 4) + 6$

12. Write the formula that defines the linear relationship represented in each of the following graphs.
   a. $y = -2(x+2)+8$ or $y = -2(x-3)-2$
   b. $b = \frac{10}{3}(a+1)-3$ or $b = \frac{10}{3}(a-2)+7$

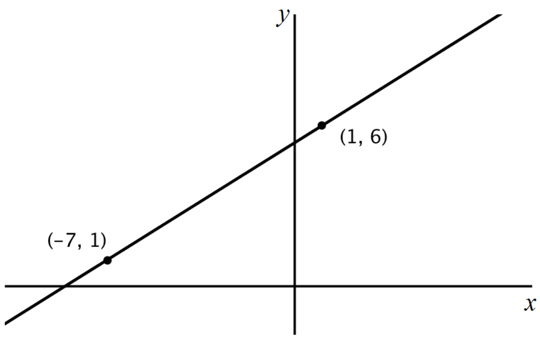


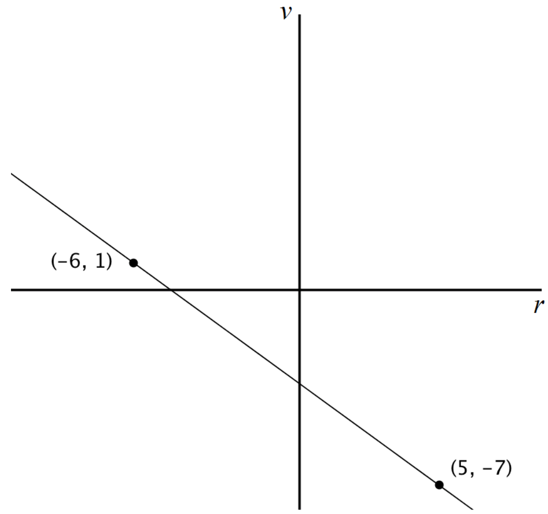



13. Write the formula that defines the linear relationship represented in each of the following graphs.

a. $y=\frac{5}{8}(x+7)+1$ or $y=\frac{5}{8}(x-1)+6$

b. $v=-\frac{8}{11}(r+6)+1$ or $v=-\frac{8}{11}(r-5)-7$




14. Write the formula that defines the linear relationship given in each of the following tables.

a. $y=-3(x-2)-8$

| ***x*** | ***y*** |
|---|---|
| –6 | 16 |
| –1 | 1 |
| 2 | –8 |
| 8 | –26 |

b. $d=2(w+4)+10$

| ***w*** | ***d*** |
|---|---|
| –9 | 0 |
| –4 | 10 |
| 1 | 20 |
| 14 | 46 |

Answers will vary. Only one example answer is provided for each.

15. Write the formula that defines the linear relationship given in each of the following tables.

a. $y=3.5(x-2)+9$

| ***x*** | ***y*** |
|---|---|
| –5 | –15.5 |
| –1 | –1.5 |
| 2 | 9 |
| 18 | 65 |

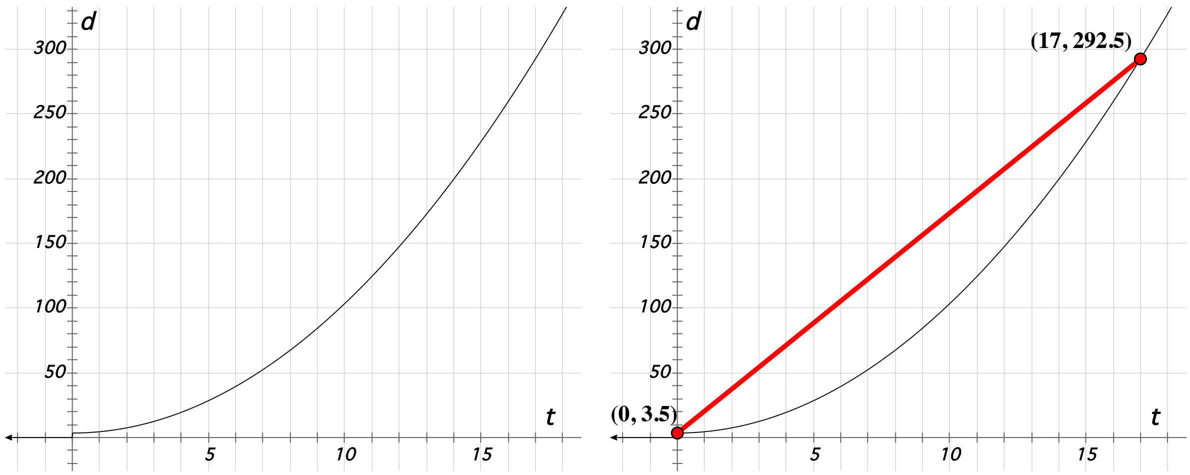
b. $d=-1.5(w-3)-5$

| ***w*** | ***d*** |
|---|---|
| –12 | 17.5 |
| –8 | 11.5 |
| 3 | –5 |
| 17 | –26 |

Answers will vary. Only one example answer is provided for each.

This investigation introduces students to the concept of average rate of change by connecting it to what they learned about constant rate of change in the previous investigation. We support students' understanding of an average rate of change as the constant rate of change needed to change a specific amount in the output quantity for a specific amount of change in the input quantity. Graphically, this constant rate of change is represented as the slope of a secant line connecting two points on the function's graph. This investigation also introduces students to the concept of the difference quotient by leveraging their past experiences in determining a linear function's constant rate of change. We have found that it is best to not emphasize the word "average" since students tend to think of all averages as arithmetic means (i.e., "add them up and divide"). We prompt students to use function notation to represent the slope of specific lines that connect two points on a function, and then describe what constant rate of change is being represented. We also introduce students to standard notation used in many calculus textbooks where the interval of change for the independent quantity is represented with $h$.

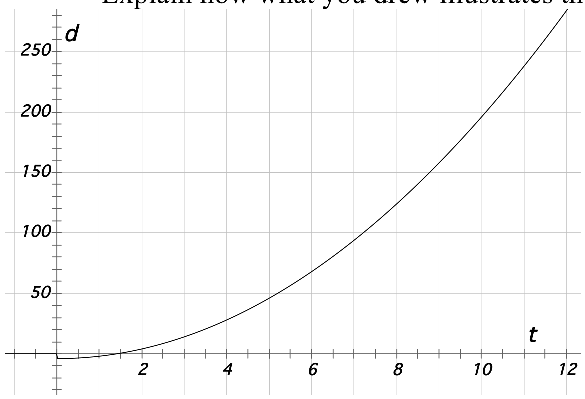
1. A car is driving away from a crosswalk. The distance $d$ (in feet) of the car from the crosswalk $t$ seconds since the car started moving is given by the formula $d = t^2 + 3.5$.
   a. Does the car's distance (in feet) from the crosswalk vary at a constant rate with respect to the number of seconds elapsed since the car started moving? Justify your response using the meaning of constant rate of change.
   Students should say more than, "$d$ is not a linear function of $t$." Push students to explain why $d$ does not vary at a constant rate with respect to $t$ by attending to the multiplicative relationship between $\Delta d$ and $\Delta t$ for various values of $\Delta t$. In other words, students should use the function definition to show that $\Delta d$ and $\Delta t$ are not proportionally related. For example, suppose $t$ changes from $t = 0$ to $t = 1$. Then the corresponding change in $d$ is given by $\Delta d = 1^2 + 3.5 - (0^2 + 3.5) = 1$. Suppose $t$ changes from $t = 1$ to $t = 2$. Then the corresponding change in $d$ is given by $\Delta d = 2^2 + 3.5 - (1^2 + 3.5) = 3$. Since $\Delta d$ is not consistent for different 1-second changes in $t$, $\Delta d$ is not proportionally related to $\Delta t$. Therefore, the car's distance (in feet) from the crosswalk does not vary at a constant rate with respect to the number of seconds elapsed since the car started moving.
   b. A second car traveling at a constant rate passed the first car the moment it started moving (at $t = 0$). The first car passed the second car 17 seconds later.
      i. At what constant speed was the second car traveling?
      We know that both cars were $0^2 + 3.5$ feet from the crosswalk the moment the first car started moving (i.e., when $t = 0$). We also know that both cars were $17^2 + 3.5$ feet from the crosswalk 17 seconds after the first car started moving (i.e., when $t = 17$). Therefore, the constant speed of the second car is given by
      $$\frac{(17^2 + 3.5) - (0^2 + 3.5)}{17 - 0} = \frac{289}{17} = 17 \text{ feet/sec.}$$
      ii. Below is a graph of the relationship between the first car's distance $d$ (in feet) from the crosswalk and the number of seconds $t$ elapsed since the car started moving. Illustrate on this graph the constant speed of the second car computed in Part (i) from $t = 0$ to $t = 17$. Explain how what you drew illustrates the second car's constant rate of change.

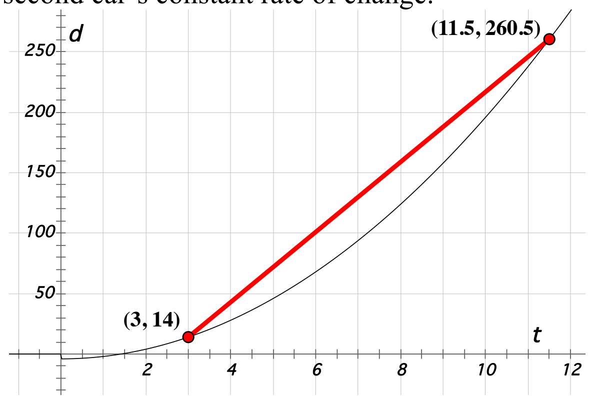



The slope of the red line connecting the points (0, 3.5) and (17, 292.5) represents the constant speed of the second car. The $x$-coordinates of the points on this red line represent the number of seconds elapsed since the second car passed the first car (i.e., since the first car started moving) and the $y$-coordinates of points on this red line represent the second car's distance from the crosswalk (in feet). The slope of this red line represents the constant speed of the second car since, for any point on this red line, the change in distance from $d = 3.5$ is 17 times as large as the corresponding change in time from $t = 0$. In other words, the changes in the second car's distance from the crosswalk (in feet) are proportionally related to the corresponding number of seconds elapsed since the second car passed the first car.

2. A car is driving through an intersection after having stopped at a red light. The distance $d$ (in feet) of the car north of the intersection $t$ seconds after it started moving is given by the formula $d = 2t^2 - 4$.
   a. Does the car's distance (in feet) north the intersection vary at a constant rate with respect to the number of seconds elapsed since the car started moving? Justify your response using the meaning of constant rate of change.
   Students should say more than, "$d$ is not a linear function of $t$." Push students to explain why $d$ does not vary at a constant rate with respect to $t$ by attending to the multiplicative relationship between $\Delta d$ and $\Delta t$ for various values of $\Delta t$. In other words, students should use the function definition to show that $\Delta d$ and $\Delta t$ are not proportionally related. For example, suppose $t$ changes from $t = 0$ to $t = 1$. Then the corresponding change in $d$ is given by $\Delta d = 2(1)^2 - 4 - (2(0)^2 - 4) = 2$. Suppose $t$ changes from $t = 1$ to $t = 2$. Then the corresponding change in $d$ is given by $\Delta d = 2(2)^2 - 4 - (2(1)^2 - 4) = 6$. Since $\Delta d$ is not consistent for different 1-second changes in $t$, $\Delta d$ is not proportionally related to $\Delta t$. Therefore, the car's distance (in feet) north of the intersection does not vary at a constant rate with respect to the number of seconds elapsed since the car started moving.
   b. A second car traveling at a constant rate passed the first car 3 seconds after it started moving. The first car passed the second car 11.5 seconds after the first car started moving.
      i. At what constant speed was the second car traveling?
      We know that both cars were $2(3)^2 - 4$ feet north of the intersection three seconds after the first car started moving (i.e., when $t = 3$). We also know that both cars were $2(11.5)^2 - 4$ feet north of the intersection 11.5 seconds after the first car started moving (i.e., when $t = 11.5$). Therefore, the constant speed of the second car is given by
$$\frac{(2(11.5)^2 - 4) - (2(3)^2 - 4)}{11.5 - 3} = \frac{246.5}{8.5} = 29 \text{ feet/sec.}$$

ii. Below is a graph of the relationship between the first car's distance $d$ (in feet) north of the intersection and the number of seconds $t$ elapsed since the car started moving. Illustrate on this graph the constant speed of the second car computed in Part (i) from $t = 3$ to $t = 11.5$. Explain how what you drew illustrates the second car's constant rate of change.




The slope of the red line connecting the points (3, 14) and (11.5, 260.5) represents the constant speed of the second car. The $x$-coordinates of the points on this red line represent the number of seconds elapsed since the second car passed the first car (i.e., three seconds after the first car started moving) and the $y$-coordinates of points on this red line represent the second car's distance north of the intersection (in feet). The slope of this red line represents the constant speed of the second car since, for any point on this red line, the change in distance from $d = 14$ is 29 times as large as the corresponding change in time from $t = 3$. In other words, the changes in the second car's distance north of the intersection (in feet) are proportionally related to the corresponding number of seconds elapsed since the second car passed the first car.

3. While running a road race, Alima's distance $d$ (in miles) from the start line $t$ minutes after she passed the start line is given by the formula $d = 0.3t^{0.7}$.

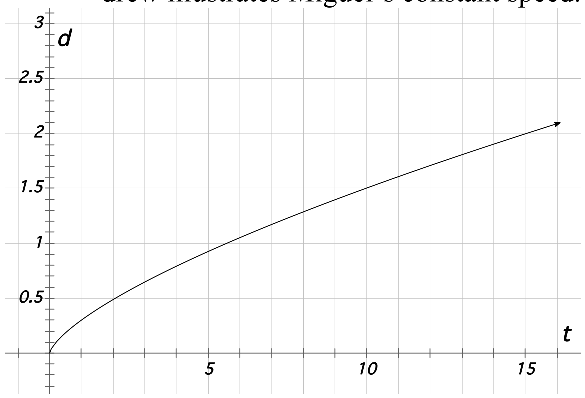
   a. Does Alima's distance (in miles) from the start line vary at a constant rate with respect to the number of minutes elapsed since she passed the start line? Justify your response using the meaning of constant rate of change.
   Students should say more than, "$d$ is not a linear function of $t$." Push students to explain why $d$ does not vary at a constant rate with respect to $t$ by attending to the multiplicative relationship between $\Delta d$ and $\Delta t$ for various values of $\Delta t$. In other words, students should use the function definition to show that $\Delta d$ and $\Delta t$ are not proportionally related. For example, suppose $t$ changes from $t = 3$ to $t = 5$. Then the corresponding change in $d$ is given by $\Delta d = 0.3(5)^{0.7} - 0.3(3)^{0.7} \approx 0.28$. Suppose $t$ changes from $t = 4$ to $t = 9$. Then the corresponding change in $d$ is given by $\Delta d = 0.3(9)^{0.7} - 0.3(4)^{0.7} \approx 0.60$. Since $\Delta d$ is not the same number of times as large as $\Delta t$, $\Delta d$ is not proportionally related to $\Delta t$. Therefore, Alima's distance (in miles) from the start line does not vary at a constant rate with respect to the number of minutes elapsed since she passed the start line.

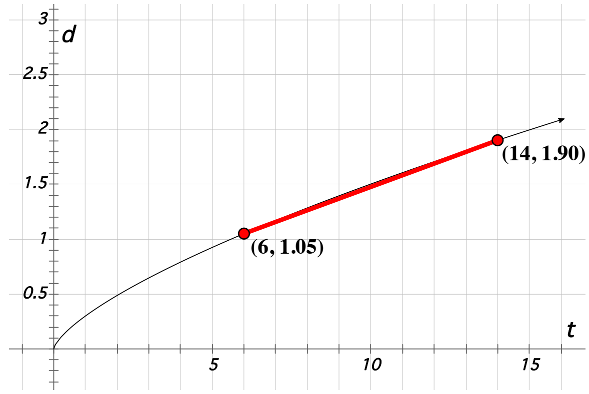
   b. Six minutes after Alima passed the start line, she passed Miguel who was running at a constant speed. Eight minutes later, Miguel passed Alima.

      i. At what constant speed was Miguel running?
      We know that both Alima and Miguel were $0.3(6)^{0.7}$ miles from the start line six minutes after Alima passed the start line (i.e., when $t = 6$). We also know that both Alima and Miguel were $0.3(14)^{0.7}$ miles from the start line 14 minutes after Alima passed the start line (i.e., when $t = 14$). Therefore, Miguel's constant speed is given by

$$\frac{0.3(14)^{0.7}-0.3(6)^{0.7}}{14-6}\approx 0.106 \text{ mi/min.}$$

ii. Below is a graph of the relationship between Alima's distance $d$ (in miles) from the start line and the number of minutes $t$ elapsed since she passed the start line. Illustrate on this graph Miguel's constant speed you computed in Part (i) from $t = 6$ to $t = 14$. Explain how what you drew illustrates Miguel's constant speed.




The slope of the red line connecting the points (6, 1.05) and (14, 1.90) represents the constant rate at which Miguel runs during the 8-minute interval between $t = 6$ and $t = 14$. The $x$-coordinates of the points on this red line represent the number of minutes elapsed since Alima passed the start line and the $y$-coordinates of points on this red line represent the Miguel's distance from the start line (in miles). The slope of this red line represents Miguel's constant speed since, for any point on this red line, the change in distance from $d = 1.05$ is approximately 0.106 times as large as the corresponding change in time from $t = 6$. In other words, the changes in Miguel's distance from the starting line (in miles) are proportionally related to the corresponding number of minutes elapsed since Alima passed the start line.

4. Let $f(x) = -x^2 + 7$.

a. Does $f(x)$ vary at a constant rate with respect to $x$? Justify your response using the meaning of constant rate of change.

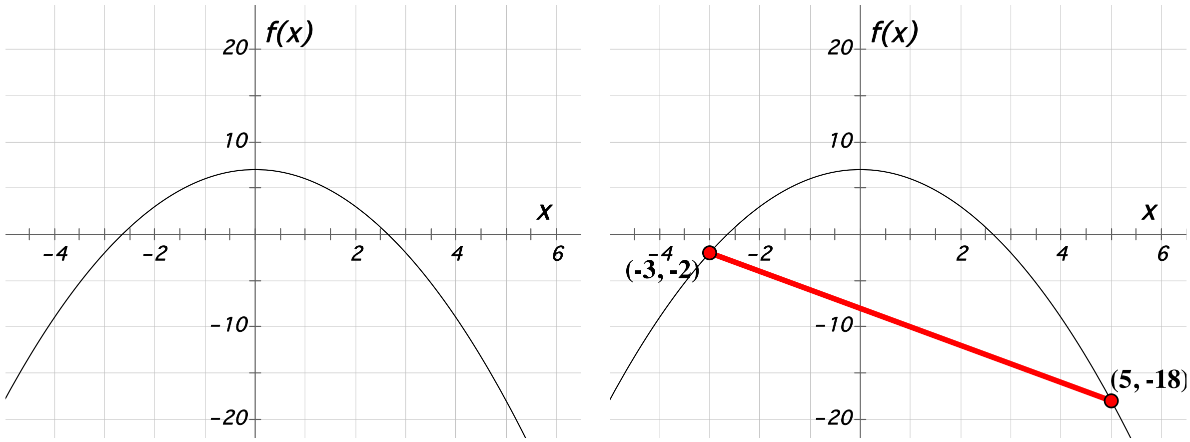
Students should say more than, "$f$ is not a linear function of $x$." Push students to explain why $f(x)$ does not vary at a constant rate with respect to $x$ by attending to the multiplicative relationship between $\Delta f(x)$ and $\Delta x$ for various values of $\Delta x$. In other words, students should use the function definition to show that $\Delta f(x)$ and $\Delta x$ are not proportionally related. For example, suppose $x$ changes from $x = 0$ to $x = 2$. Then the corresponding change in $f(x)$ is given by $\Delta f(x) = -(2)^2 + 7 - (-(0)^2 + 7) = -4$. Suppose $x$ changes from $t = 1$ to $t = 5$. Then the corresponding change in $f(x)$ is given by $\Delta f(x) = -(5)^2 + 7 - (-(1)^2 + 7) = -12$. Since $\Delta f(x)$ is not the same number of times as large as $\Delta x$, $\Delta f(x)$ is not proportionally related to $\Delta x$. Therefore, $f(x)$ does not vary at a constant rate with respect to $x$.

b. i. What is the constant rate of change of the linear function $g$ that has the same change in output values over the interval $x = -3$ to $x = 5$ as the function $f$?

The constant rate of the linear function $g$ that has the same change in output values over the interval $x = -3$ to $x = 5$ as the function $f$ is given by

$$\frac{f(5)-f(-3)}{5-(-3)}=\frac{-(5)^2+7-(-(-3)^2+7)}{5-(-3)}=\frac{-18-(-2)}{5+3}=-2.$$

ii. Below is a graph of $f$. Illustrate on this graph the constant rate of change you computed in Part (i) from $x = -3$ to $x = 5$. Explain how what you drew illustrates a constant rate of change.




The slope of the red line connecting the points (–3, –2) and (5, –18) represents the constant rate of change of a linear function $g$ that has the same change in output as the function $f$ over the interval [–3, 5]. For any point on this red line, the change in $g(x)$ from $g(x) = -2$ is –2 times as large as the corresponding change in $x$ from $x = -3$. In other words, the changes in $x$ from $x = -3$ are proportionally related to the corresponding change in $g(x)$.

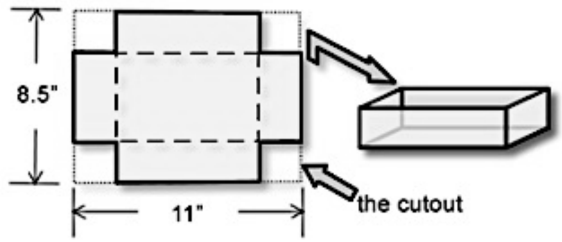
In Part (b) of Problems 1-4, you computed what is called an ***average rate of change***. The average rate of change of a function $f$ from $x = x_1$ to $x = x_2$ is the constant rate of change of a linear function $g$ that has the same change in output as the function $f$ over the interval $[x_1, x_2]$.

The function $g$ has the same change in output as the function $f$ from $x = x_1$ to $x = x_2$ if $f(x_1) = g(x_1)$ and $f(x_2) = g(x_2)$. The *average rate of change* of $f$ over the interval $[x_1, x_2]$ is the constant rate of change $\dfrac{g(x_2) - g(x_1)}{x_2 - x_1}$ of the linear function $g$.

5. Use the definition of average rate of change above to explain why the values you computed in response to Part (b) of Problems 1-4 are average rates of change.
In Part (b) of Problem 1, I computed the constant speed that a second car would have to travel to cover the same distance as the first car during the 17 seconds after the first car started moving.
In Part (b) of Problem 2, I computed the constant speed that a second car would have to travel to cover the same distance as the first car during the 8.5-second interval between $t = 3$ and $t = 11.5$.
In Part (b) of Problem 3, I computed the constant speed that a second runner (Miguel) would have to run to cover the same distance as Alima during the 8-minute interval between $t = 6$ and $t = 14$.
In Part (b) of Problem 4, I computed the constant rate of change of a linear function that achieves the same change in output as the function $f$ over the interval [–3, 5].

6. Write an expression that represents the average rate of change of the function $f$ over the interval $[x_1, x_2]$.
$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

7. An open-top box is created by cutting squares out of the corners of an 8.5-inch by 11-inch sheet of paper and then folding up the sides (see image below).




a. Define a function $f$ to determine the volume of the box (measured in cubic inches) in terms of the length $x$ of the side of the square cutout (in inches).
$f(x) = x(11 - 2x)(8.5 - 2x)$.

b. Describe the meaning of each of the following expressions in the context of the situation:
i. $f(x+3)$ ii. $f(x+3) - f(x)$ iii. $\frac{f(x+3)-f(x)}{(x+3)-x}$

i. This expression represents the volume of the box for a cutout length of $x + 3$ inches.
ii. This expression represents the difference in volume of the box for a cutout length of $x + 3$ inches and for volume of the box for a cutout length of $x$ inches.
iii. This expression represents the average rate of change of the volume of a box for an increase of 3 inches of the cutout length from $x$ to $x + 3$ inches.

c. Evaluate $\frac{f(x+3)-f(x)}{(x+3)-x}$ for $x = 0.5$. Describe the meaning of this value in the context of the situation.

$$\begin{aligned}\frac{f(x+3)-f(x)}{(x+3)-x} &= \frac{f(0.5+3)-f(0.5)}{(0.5+3)-0.5}\\ &= \frac{f(3.5)-f(0.5)}{3}\\ &= \frac{\left[(3.5)(11-2(3.5))(8.5-2(3.5))\right]-\left[(0.5)(11-2(0.5))(8.5-2(0.5))\right]}{3}\\ &= \frac{[21]-[37.5]}{3}\\ &= -5.5\end{aligned}$$

The value –5.5 represents the constant rate of change that would achieve the same change in the volume of a box (as what was actually achieved) for an increase of the length of the side of the cutout from 0.5 inches to 3.5 inches. This constant rate of change is –5.5 cubic inches per inch of length of the side of the cutout, and is referred to as the average rate of change over the interval.

8. The area $A$ (measured in square feet) of a circular oil slick as a function of the amount of time (measured in minutes) since the oil leak started is given by the function $g(t) = \pi(7.84t)^2$.

a. Describe the meaning of $\frac{g(t+5)-g(t)}{5}$ (simplified from $\frac{g(t+5)-g(t)}{(t+5)-t}$) in the context of this situation.

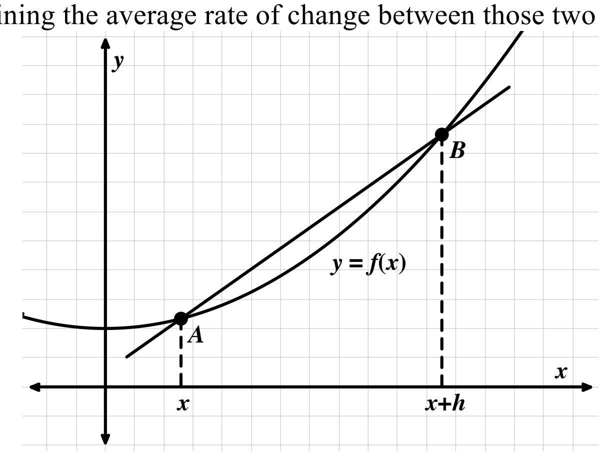
The expression represents the average rate of change of the area of an oil slick for an increase of time of 5 minutes.

b. Evaluate $\frac{g(t+5)-g(t)}{5}$ when $t = 1.5$. Describe the meaning of this value.

$$\begin{aligned}\frac{g(t+5)-g(t)}{5} &= \frac{g(1.5+5)-g(1.5)}{5}\\ &= \frac{g(6.5)-g(1.5)}{5}\\ &= \frac{\left[7.84\pi(6.5)^2\right]-\left[7.84\pi(1.5)^2\right]}{5}\\ &= \frac{[1040.62]-[55.42]}{5}\\ &\approx 197.04\end{aligned}$$

The value 197.04 ft²/min represents the average rate of change of the area of the oil slick (with respect to time) as the time increases from 1.5 to 6.5 minutes. Thus, the area of the slick would have had to increase at a constant rate of 197.04 $ft^2$ per minute from 1.5 to 6.5 minutes to result in the same change in area.

In general, the expression $\frac{f(x+h)-f(x)}{h}$ (simplified from $\frac{f(x+h)-f(x)}{(x+h)-x}$) where $h$ represents the change in $x$ is called the ***difference quotient***. The difference quotient is the average rate of change for a function between two input-output pairs (see Point $A$ to Point $B$ on the graph below). Using function notation, we say that you can find the average rate of change between any two points $(x, f(x))$ and $(x+h, f(x+h))$ by computing: $\frac{\Delta f(x)}{\Delta x} = \frac{f(x+h)-f(x)}{(x+h)-x} = \frac{f(x+h)-f(x)}{h}$. Thus, when computing the difference quotient for two points on a function, you are determining the average rate of change between those two input-output pairs.



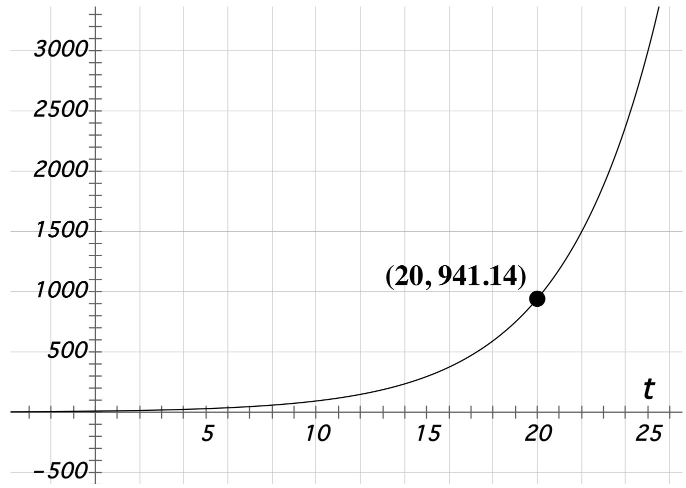

9. The function $j$ defined by the formula $j(t) = 1{,}645(1.06)^t$ determines the population of Telluride, Colorado $t$ years since January 1, 1990. Define a function $g$ that determines the average rate of change of Telluride's population over any 0.2-year interval since January 1, 1990.

$$g(t) = \frac{j(t+0.2)-j(t)}{(t+0.2)-t} = \frac{1{,}645(1.06)^{t+0.2} - 1{,}645(1.06)^t}{0.2}.$$

10. The function $g$ defined by the formula $g(t) = 2t^2 + 4t$ determines a car's distance from a stop sign $t$ seconds after it passed the stop sign. Define a function $k$ that determines the car's average speed over any 0.5-second interval since the car passed the stop sign.

$$k(t) = \frac{g(t+0.5)-g(t)}{(t+0.5)-t} = \frac{2(t+0.5)^2+4(t+0.5)-(2t^2+4t)}{0.5}.$$

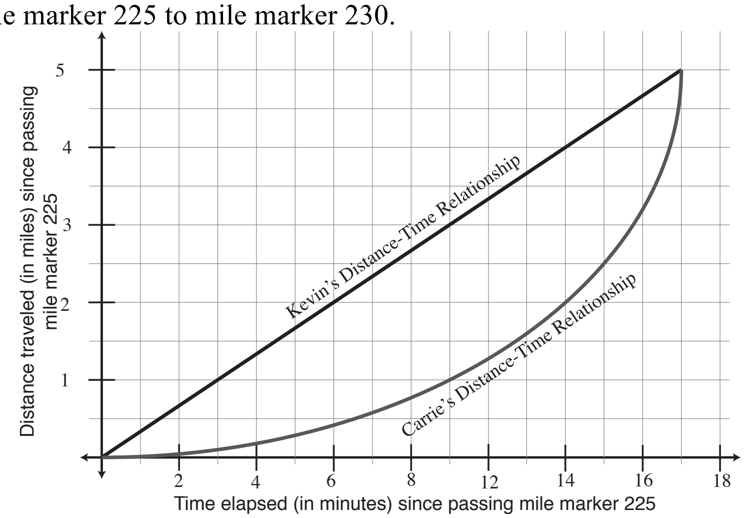
11. The function $f$ defines the relationship between the value (in dollars) of a Picasso painting $t$ years since it was created in 1932. The function $g$ defined by the formula $g(t) = \frac{f(t+1.2)-f(t)}{(t+1.2)-t}$ is graphed below. Explain what the selected point on the graph of the function $g$ represents.




The point (20, 941.14) means that, even though the value of Picasso's painting does not increase at a constant rate, the constant rate of appreciation of $941.14 per year yields the same change in value of Picasso's painting during the time interval from 20 years to 21.2 years after he painted it.

**Homework**

1. The following graph represents the distance-time relationship for Kevin and Carrie as they cycled on a road from mile marker 225 to mile marker 230.




a. How does the distance traveled and time elapsed compare for Carrie and Kevin as they traveled from mile marker 225 to mile marker 230?
Both Carrie and Kevin traveled the same distance (5 miles) and they took the same amount of time (17 minutes) to travel those 5 miles.

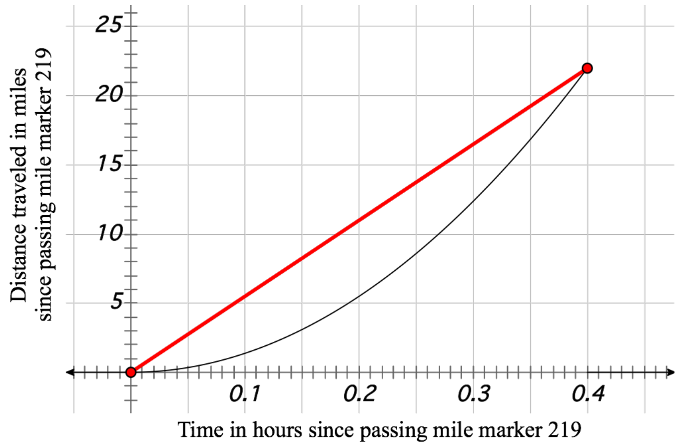
b. How do Carrie and Kevin's speeds compare as they travel from mile marker 225 to mile marker 230?

Kevin's speed is constant for the entire 17-minute period, Carrie traveled slower in the beginning and gradually sped up.

c. How do Carrie and Kevin's average speeds compare over the time interval as they traveled from mile marker 225 to mile marker 230?
Their average speeds were the same (5/17 miles per minute).

2. When running a marathon you heard the timer call out 12 minutes as you passed mile-marker 2.
   a. What quantities could you measure to determine your speed as you ran the race? Define variables to represent the quantities' values and state the units you will use to measure the value of each of these quantities.
   Let $d$ = distance traveled in miles.
   Let $t$ = the number of minutes elapsed since you started the race.
   b. As you passed mile-marker 5 you heard the timer call out 33 minutes. What was your average speed from mile 2 to mile 5?
   Average speed: $\frac{\Delta d}{\Delta t} = \frac{5-2}{33-12} = \frac{3}{21} = \frac{1}{7}\frac{\text{miles}}{\text{min}}$
   c. Assume that you continued running at the same constant speed as computed in Part (b) above. How much distance did you cover as your time spent running increased from 35 minutes after the start of the race to 40 minutes after the start of the race?
   For every one minute, the distance traveled changes by 1/7 of a mile, so for when the amount of time elapsed since the start of the race changes by 5 minutes, the corresponding change in the distance traveled is 5/7 miles.
   d. If you passed mile marker 5 at 33 minutes, what average speed do you need to run for the remainder of the race to meet your goal to complete the 26.2-mile marathon in 175 minutes?
   You have 175 – 33 = 142 minutes so that you can complete the race in a total of 175 minutes, and to complete the race you have 26.2 – 5 = 212.2 miles remaining. Thus, your average speed for the rest of the race should be (21.2/142) = 0.149 miles per minute.
   e. What is the meaning of average speed in this context?
   If you were tied with another runner at mile-marker 5, and that runner ran the rest of the race at a constant speed of 0.149 miles per minute then you would need to tie with him at the finish line to meet your goal.

3. On a trip from Tucson to Phoenix via Interstate 10, you used your cruise control to travel at a constant speed for the entire trip. Since your speedometer was broken, you decided to use your watch and the mile markers to determine your speed. At mile marker 219 you noticed that the time on your digital watch just advanced to 9:22 AM. At mile marker 197 your digital watch advanced to 9:46 AM.
   a. Compute the constant speed at which you traveled over the time period from 9:22 AM to 9:46 AM.
   $\frac{|197-219|}{\left(\frac{24}{60}\right)} = 55$ miles per hour
   b. As you were passing mile marker 219 you also passed a truck. The same truck sped by you at exactly mile marker 197.
      i. Construct a distance-time graph of your car. On the same graph, construct one possible distance-time graph for the truck. Be sure to label the axes.



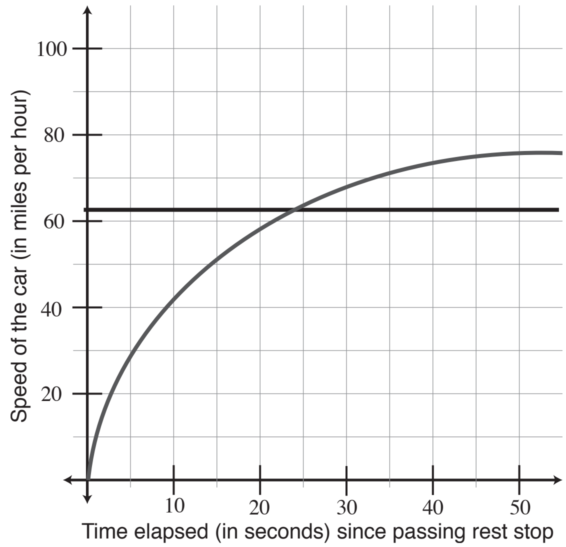

ii. Compare the speed of the truck to the speed of the car between 9:22 AM and 9:45 AM.
The car was traveling faster (the car covered more distance than the truck in the same amount of time) at the beginning of the interval and the truck was traveling faster as the truck passed the car.

iii. Compare the distance that your car traveled over this part of the trip with the distance that the truck traveled over this same part of the trip. Compare the time that it took the truck to travel this distance with the time that it took your car to travel this distance. What do you notice?
The car and the truck both traveled the same distance in the same amount of time even though they were not next to each other and traveling the same speed the entire trip.

iv. Why are the average speed of the car and the average speed of the truck the same?
The average speed measures the constant rate at which an object would need to travel in order to produce the same net change in distance in an identical segment of time. Since we only measured the position of the truck and car at the ends of the segment, and they were next to each other at those points and both had traveled the same amount of time, the average speeds are the same.

v. Phoenix is another 53 miles past mile marker 197. Assuming you continued at the constant speed, at what time should you arrive in Phoenix?
Assuming that you continue to travel at 55 miles per hour, you will need to travel (53/55) of an hour past 9:46 AM. Thus it will take (60)(53/55) = 57.82 minutes after 9:46 AM to arrive in Phoenix, a time of about 10:44 AM.

4. The graph that follows represents the speeds of two cars (Car A and Car B) in terms of the elapsed time in seconds since being at a rest stop. Car A is traveling at a constant speed of 62 miles per hour. As Car A passes the rest stop, Car B pulls out beside Car A and they both continue traveling down the highway.




a. Which graph represents Car A's speed and which graph represents Car B's speed? Explain.
Since this is a speed versus time graph, a constant speed of 65 mph is represented by a horizontal line at $y = 6.5$. So the horizontal line represents Car A's speed. Car B's speed increases from 0 mph at $t = 0$, which is represented by the non-horizontal graph. This graph begins at the origin and the speed of the car increases as the number of seconds since passing the rest stop increases.

b. Which car is further down the road 20 seconds after being at the rest stop? Explain.
Twenty seconds after passing the rest stop, Car A is further down the road because during the entire 20-second interval Car A was traveling at a speed greater than that of Car B.

c. Explain the meaning of the intersection point.
The intersection point represents the number of seconds that have elapsed since passing the rest stop when the two cars are traveling the same speed, 62 mph.

d. What is the relationship between the positions of Car A and Car B 27 seconds after being at the rest stop?
Twenty-seven seconds after being at the rest stop, the speed of Car B is just slightly greater than the speed of Car A. However, this does not imply that Car B is ahead of Car A.

e. Car B catches up with Car A 64.5 seconds after Car A and Car B passed the rest stop. What is the average speed of Car B over the interval from 0 seconds to 64.5 seconds after leaving the rest stop? Explain.
Since Car A and Car B are in the same position 64.5 seconds after both cars past the rest stop, the average speed of Car B over this interval is the constant speed of Car A, or 62 miles per hour.

**Instructions for Problems 5-14**: Let $d$ be the distance of a car (in feet) from mile marker 420 on a country road and let $t$ be the time elapsed (in seconds) since the car passed mile marker 420. The formulas below represent various ways these quantities might be related. For each of the following:

i. Determine the average speed of the car using the given formula and the specified time interval.
ii. Explain the meaning of average speed in the context of this situation.

5. $d = t^2$ from $t = 5$ to $t = 30$.

i.

$d_1 = 5^2 = 25$

$d_2 = 30^2 = 900$ Average Speed $= \frac{900-25}{30-5} = 35$ feet per second

ii. The average speed of 35 ft/sec represents constant speed another vehicle would have to travel to cover that same distance (875 feet) in the same amount of time (25 seconds).

6. $d = -3(-19t - 1)$ from $t = 3$ to $t = 9$.
   i.
   $d_1 = -3(-19(3) - 1) = 174$
   $d_2 = -3(-19(9) - 1) = 516$  Average Speed $= \frac{516-174}{9-3} = 57$ feet per second
   ii. The average speed of 57 ft/sec represents the constant speed the vehicle would have to travel to cover that same distance (342 feet) in the same amount of time (6 seconds).

7. $d = 5(12t + 1) + 3t$ from $t = 0.5$ to $t = 3.75$.
   i.
   $d_1 = 5(12(.5) + 1) + 3(0.5) = 36.5$
   $d_2 = 5(12(3.75) + 1) + 3(3.75) = 241.25$  Average Speed $= \frac{241.24-36.5}{3.75-0.5} = 63$ feet per second
   ii. The average speed of 63 ft/sec represents the constant speed the vehicle would have to travel to cover that same distance (204.75 feet) in the same amount of time (3.25 seconds).

8. $d = \frac{10t(t+5)-14}{2}$ from $t = 0$ to $t = 5$.
   i.
   $d_1 = \frac{10(0)(0+5)-14}{2} = -7$
   $d_2 = \frac{10(5)(5+5)-14}{2} = 243$  Average Speed $= \frac{243-(-7)}{5-0} = 50$ feet per second
   ii. The average speed of 50 ft/sec represents the constant speed the vehicle would have to travel to cover that same distance (250 feet) in the same amount of time (5 seconds).

9. $d = \frac{1}{3}\left(9t^2 + 155t - (11t - 6)\right)$ from $t = 2$ to $t = 4$.
   i.
   $d_1 = \frac{1}{3}(9(2)^2 + 155(2) - (11(2) - 6)) = 110$
   $d_2 = \frac{1}{3}(9(4)^2 + 155(4) - (11(4) - 6)) = 242$  Average Speed $= \frac{242-110}{4-2} = 66$ feet per second
   ii. The average speed of 66 ft/sec represents the constant speed the vehicle would have to travel to cover that same distance (132 feet) in the same amount of time (2 seconds).

10. $d = (2t + 7)(3t - 2)$ from $t = 2$ to $t = 2.75$.
   i.
   $d_1 = (2(2) + 7)(3(2) - 2) = 44$
   $d_2 = (2(2.75) + 7)(3(2.75) - 2) = 78.125$  Average Speed $= \frac{78.125-44}{2.75-2} = 45.5$ feet per second
   ii. The average speed of 45.5 ft/sec represents the constant speed the vehicle would have to travel to cover that same distance (34.125 feet) in the same amount of time (0.75 seconds).

11. $d = \left(\frac{1}{3}t + 60\right)\left(t + \frac{1}{2}\right)$ from $t = 1$ to $t = 4$.
   i.
   $d_1 = \left(\frac{1}{3}(1) + 60\right)\left(1 + \frac{1}{2}\right) = 90.5$
   $d_2 = \left(\frac{1}{3}(4) + 60\right)\left(4 + \frac{1}{2}\right) = 276$  Average Speed $= \frac{276-90.5}{4-1} \approx 61.833$ feet per second
   ii. The average speed of 61.833 ft/sec represents the constant speed the vehicle would have to travel to cover that same distance (185.5 feet) in the same amount of time (3 seconds).

12. $d=(t+6)(t+3)+7t-20+11t-\frac{7}{8}t^2$ from $t = 30$ to $t = 35$.
   i.
   $$d_1=(30+6)(30+3)+7(30)-20+11(30)-\tfrac{7}{8}(30)^2=920.5$$
   $$d_2=(35+6)(35+3)+7(35)-20+11(35)-\tfrac{7}{8}(35)^2=1096.125$$
   Average Speed $=\frac{1096.125-920.5}{35-30}=35.125$ feet per second
   ii. The average speed of 35.125 ft/sec represents the constant speed the vehicle would have to travel to cover that same distance (175.625 feet) in the same amount of time (5 seconds).

13. $d=\frac{1}{8}t\left(3t^2+1.5t\right)+16t-3$ from $t = 2$ to $t = 4$.
   i.
   $$d_1=\tfrac{1}{8}(2)\left(3(2)^2+1.5(2)\right)+16(2)-3=39$$
   $$d_2=\tfrac{1}{8}(4)\left(3(4)^2+1.5(4)\right)+16(4)-3=133$$
   Average Speed $=\frac{133-39}{4-2}=47$ feet per second
   ii. The average speed of 47 ft/sec represents the constant speed the vehicle would have to travel to cover that same distance (94 feet) in the same amount of time (2 seconds).

14. $d=\dfrac{t\left(\frac{1}{2}t^2+15\right)+3t\left(\frac{1}{10}t^2+\frac{4}{3t}\right)}{5}$ from $t = 5$ to $t = 9$.
   i.
   $$d_1=\frac{5\left(\frac{1}{2}(5)^2+15\right)+3(5)\left(\frac{1}{10}(5)^2+\frac{4}{3(5)}\right)}{5}=35.8$$
   $$d_2=\frac{9\left(\frac{1}{2}(9)^2+15\right)+3(9)\left(\frac{1}{10}(9)^2+\frac{4}{3(9)}\right)}{5}=144.44$$
   Average Speed $=\frac{144.44-35.8}{9-5}=27.16$ feet per second
   ii. The average speed of 27.16 ft/sec represents the constant speed the vehicle would have to travel to cover that same distance (108.64 feet) in the same amount of time (4 seconds).

15. Consider the function $f$ defined by $f(x) = x^2 - 6x + 10$ that represents the altitude of a U.S. Air Force test plane (in thousands of feet) during a recent test flight as a function of elapsed time (in minutes) since being released from its airborne launcher.
   a. Find the average rate of change of the plane's altitude with respect to time as the time varies from $x = 2$ minutes to $x = 2.1$ minutes. Show your work.
   $$\frac{\Delta f}{\Delta t}=\frac{f_2-f_1}{t_2-t_1}=\frac{f(2.1)-f(2)}{2.1-2}=\frac{\left(2.1^2-6(2.1)+10\right)-\left(2^2-6(2)+10\right)}{0.1}=\frac{1.81-2}{0.1}=-1.9$$
   The average rate of change of the plane's altitude with respect to time is –1,900 ft/min.
   b. What is the meaning of the *average rate of change* you determined in Part (a) in this context?
   The average rate of change calculated above represents the constant rate (in thousands of feet per minute) at which the test plane would have to ascend (or descend, in this case, because the average rate of change is negative) to have the same change in altitude (–190 feet) and the same change in time (0.1 minutes) as the plane's actual test flight.
   c. Explain the meaning of the expression $f(k + 0.1)$.
   The expression $f(k + 0.1)$ represents the altitude of a U.S. Air Force test plane (in thousands of feet) $k + 0.1$ minutes after the plane was released from its airborne launcher.

d. Explain the meaning of the expression $f(k + 0.1) - f(k)$.
This expression represents the change in the two altitudes of a U.S. Air Force test plane: one altitude $k + 0.1$ minutes since being released from its airborne launcher and another altitude $k$ minutes since being released from its airborne launcher.
Another way to interpret the expression is that $f(k + 0.1) - f(k)$ represents the change in altitude of a U.S. Air Force test plane as time increased from $k$ minutes since being released from its airborne launcher to $(k + 0.1)$ minutes since being released from its airborne launcher.

f. What does the expression $\dfrac{f(k+0.1)-f(k)}{(k+0.1)-k}$ represent in the context of this problem?

The expression $\dfrac{f(k+0.1)-f(k)}{(k+0.1)-k}$ represents the average rate of change of $f$ between an input of "$k$" and "$k + 0.1$". In the context of the problem, $\dfrac{f(k+0.1)-f(k)}{(k+0.1)-k}$ represents the average rate of change of the plane's altitude with respect to time, as the time varies from $k$ minutes to $k + 0.1$ minutes.

16. Write an expression that represents the average rate of change of the given function over an input interval of length $h$. Be sure to simplify your answer.

a. $f(x) = 12x + 6.5$

$$\frac{f(x+h)-f(x)}{h}=\frac{(12(x+h)+6.5)-(12x+6.5)}{h}=\frac{12h}{h}=12$$

It makes sense that the difference quotient is constant because $f$ is a linear function, and linear functions have a constant rate of change. Therefore, if you compute the difference quotient (i.e., average rate of change) between any two points, the result will be 12.

b. $f(x) = 97$

$$\frac{f(x+h)-f(x)}{h}=\frac{97-97}{h}=\frac{0}{h}=0$$

It makes sense that the difference quotient is zero because $f$ is a constant function, and the output of a constant function does not change in response to changes in the input value.

c. $f(x) = 6x^2 + 7x - 11$

$$\begin{aligned}\frac{f(x+h)-f(x)}{h}&=\frac{\left(6(x+h)^2+7(x+h)-11\right)-\left(6x^2+7x-11\right)}{h}\\&=\frac{12xh+7h+6h^2}{h}\\&=12x+7+6h\end{aligned}$$

It makes sense that the difference quotient is an increasing function, because the average rate of change of $f$ increases as $x$ increases.

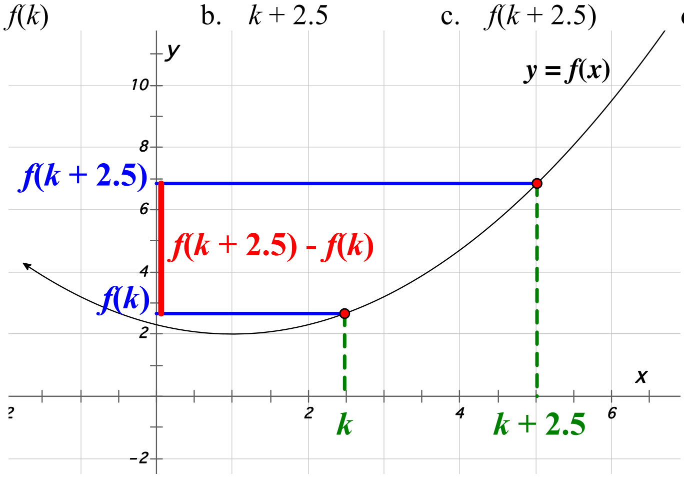
d. $f(x) = 3x^3 - 9$

$$\begin{aligned}\frac{f(x+h)-f(x)}{h}&=\frac{(3(x+h)^3-9)-(3x^3-9)}{h}\\&=\frac{9x^2h+9xh^2+3h^3}{h}\\&=9x^2+9xh+3h^2\end{aligned}$$

e. $f(x)=\dfrac{1}{2x}$

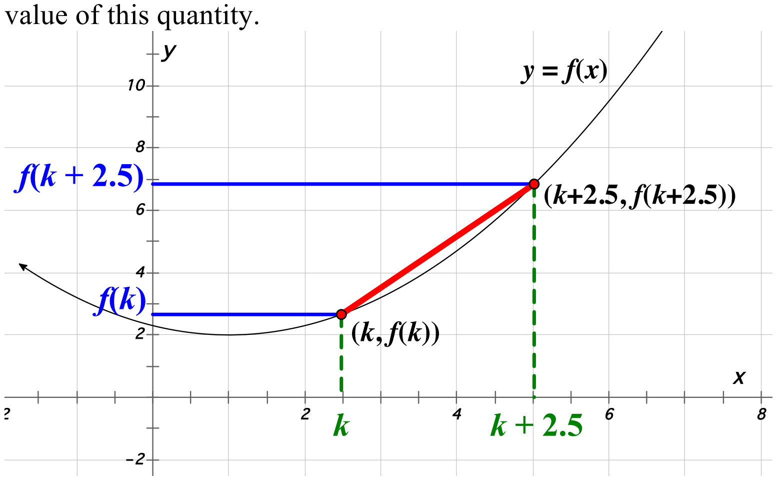
$$\frac{f(x+h)-f(x)}{h}=\frac{\frac{1}{2(x+h)}-\frac{1}{2x}}{h}$$
$$=\frac{x-(x+h)}{2xh(x+h)}$$
$$=\frac{-h}{2xh(x+h)}$$
$$=\frac{-1}{2x(x+h)}$$

17. Use the graph of $y = f(x)$ and an input value of $x = k$ to represent the following quantities on the graph.
a. $f(k)$ b. $k + 2.5$ c. $f(k + 2.5)$ d. $f(k + 2.5) - f(k)$




Note that $k$ represents an arbitrary fixed value for the input, and that $f(k + 2.5) - f(k)$ is represented by subtracting the distance of $f(k)$ from the $x$-axis from the distance that $f(k + 2.5)$ is from the $x$-axis.

e. Represent the quantity $\frac{f(k+2.5)-f(k)}{2.5}$ on the graph above. Explain how what you drew represents the value of this quantity.

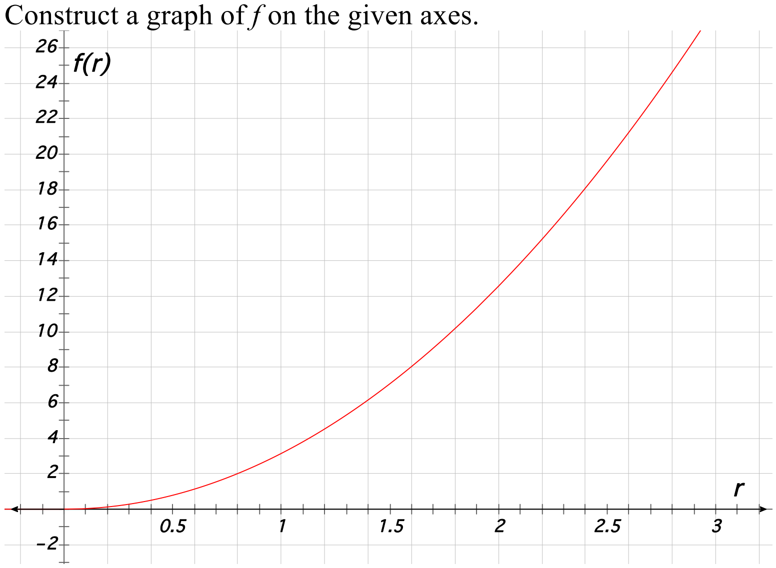



The quantity $\frac{f(k+2.5)-f(k)}{2.5}$ can be represented on the graph by thinking of the value of the slope of the line that connects the two points, $(k, f(k))$ and $(k + 2.5, f(k + 2.5))$, as shown by the red line in the graph above.

The primary purpose of this investigation is to allow students to recognize that we often make the assumption that there is a roughly linear relationship between the input and output quantities of a function on small intervals of the domain. In other words, we often assume that the output quantity varies at a constant rate with respect to the input quantity over small intervals of the input quantity. This concept of *local constant rate of change* provides the conceptual foundation for the idea of derivative, which is the focus of Investigations 4–6.

The understandings promoted in this investigation were informed by the content in Chapter 4 of *Calculus: Newton Meets Technology* by Patrick Thompson, Mark Ashbrook, Stacy Musgrave, and Fabio Milner (Thompson et al., 2015).

1. A stone is dropped into a lake creating a circular ripple that travels outward.
   a. Define a function $f$ that determines the area $f(r)$ of a circle (in square inches) in terms of the circle's radius length $r$ (in inches).
      $f(r) = \pi r^2$
   b. Construct a graph of $f$ on the given axes.




   c. Does the area of a circle increase at a constant rate of change with respect to its radius length? Explain.
      The area of the circle does *not* increase at a constant rate with respect to the length of its radius. For equal changes in the radius length the circle's area does not increase the same amount. Prompt students to justify this statement using their graph.
   d. Use $f$ to determine the average rate of change of the area of the circle (with respect to the radius) as $r$ changes from:
      i. 1.5 to 2 ii. 1.9 to 2 iii. 2 to 2.1 iv. 2 to 2.5

      i. $\dfrac{f(2)-f(1.5)}{2-1.5} = \dfrac{\pi(2)^2-\pi(1.5)^2}{2-1.5} \approx 11.00$

      ii. $\dfrac{f(2)-f(1.9)}{2-1.9} = \dfrac{\pi(2)^2-\pi(1.9)^2}{2-1.9} \approx 12.25$

      iii. $\dfrac{f(2.1)-f(2)}{2.1-2} = \dfrac{\pi(2.1)^2-\pi(2)^2}{2.1-2} \approx 12.88$

      iv. $\dfrac{f(2.5)-f(2)}{2.5-2} = \dfrac{\pi(2.5)^2-\pi(2)^2}{2.5-2} \approx 14.14$

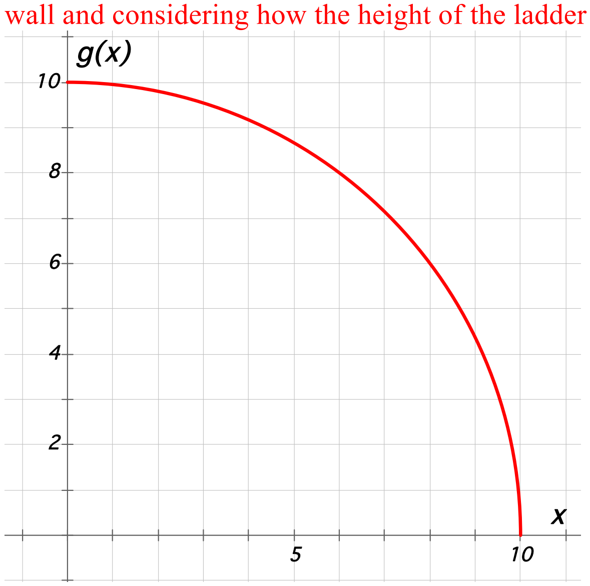
e. What do each of the average rates of change that you computed in Part (c) represent?
Each of these average rates of change represent the constant rate of change on each interval that achieves the same increase in the circle's area as what was achieved by the function on that interval.

2. A 10-foot ladder is learning vertically against a wall. David pulls the base of the ladder away from the wall at a constant rate until it is lying flat on the floor. As he does this, the top of the ladder slides down on the wall. Consider how *the height of the ladder on the wall* is related to the amount by which *the bottom of the ladder is pulled away from the wall.* Use your pen or some straight edge to simulate the ladder situation and then answer the following questions.

a. Select the statement that completes the sentence: As the base of the ladder is pulled away from the wall by successive equal amounts, the distance from the top of the ladder from the floor …
   i. changes by equal amounts.
   ii. changes less and less.
   iii. changes more and more.

As the base of the ladder is pulled away from the wall by successive equal amounts, the distance from the top of the ladder from the floor ***changes more and more***.

b. Use the thinking you used in Part (a) to draw a rough sketch of a function $g$ that defines the distance of the top of the 10-foot ladder from the floor in terms of the number of feet $x$ of the base of the 10-foot ladder from the wall.
Encourage students to consider how the two distances are changing together by considering small successive increases in the amount by which the bottom of the ladder is being pulled away from the wall and considering how the height of the ladder from the floor is changing.




c. Define a function $g$ that represent the distance (in feet) of the top of the 10-foot ladder $g(x)$ from the floor in terms of the distance (in feet) of the base of the ladder from the wall, $x$.
$g(x) = \sqrt{100 - x^2}$

d. Determine the average rate of change of the height of the top of the ladder from the floor in terms of the distance of the bottom of the ladder from the wall as $x$ increases from:
i. 0 to 1 ii. 4 to 5 iii. 9 to 10

i. $\dfrac{\sqrt{100-1^2} - \sqrt{100-0^2}}{1-0} \approx 0.050$

ii. $\dfrac{\sqrt{100-5^2}-\sqrt{100-4^2}}{5-4}\approx 0.501$

iii. $\dfrac{\sqrt{100-10^2}-\sqrt{100-9^2}}{10-9}\approx -4.359$

e. What do you notice about how the average rate of change changes as $x$ increases from 0 to 10? (Is the value of the average rate of change increasing or decreasing on this interval?)
The average rate of change of $g(x)$ with respect to $x$ decreases as $x$ increases on the interval [0, 10]. This supports the response to Part (a): "As the base of the ladder is pulled away from the wall by successive equal amounts, the distance from the top of the ladder from the floor ***changes more and more***."

3. The table below shows the estimated population for Rutherford County between July 1, 2010 and July 1, 2015. (Source: US Census Bureau, Population Division)

| **Year** | 2010 | 2011 | 2012 | 2013 | 2014 | 2015 |
|---|---|---|---|---|---|---|
| **Population** | 263,781 | 269,097 | 274,339 | 281,596 | 289,147 | 298,612 |

a. Does the population of Rutherford County vary at a constant rate with respect to the number of years elapsed since July 1, 2010? Explain.
The population of Rutherford County does not vary at a constant rate with respect to the number of years elapsed since July 1, 2010 since changes in the population are not proportionally related to changes in the number of years elapsed since July 1, 2010. For example, we notice that the population increase during the one-year interval between July 1, 2010 and July 1, 2011 (5,316) is not equal to the population increase during the one-year interval between July 1, 2011 and July 1, 2012 (5,242). The changes in the population of Rutherford County are therefore not related by a constant multiple to the changes in the number of years elapsed since July 1, 2010, which implies that these quantities do not vary at a constant rate with respect to each other.

b. What strategy would you use to estimate the population of Rutherford County on March 1, 2012? Think carefully, write down your strategy, and then use it to estimate the population. (Keep in mind there are many valid approaches.)
Part (b) could be assigned as a homework exercise before students begin working on the investigation in class. It is best, however, to allow students time to work on this problem in groups. They will need about ten minutes to devise, describe, and implement their strategies. This problem is designed to discourage students from trying to plot the points and construct a continuous graph. (Of course, this is not an incorrect approach, but taking this approach circumvents the meanings we designed this task to support.) The purpose of this activity is to provide an occasion for students to assume some type of *local linear* behavior from the function, and the instructor should get as many examples of this assumption on display as possible. Some groups will assume the output remains constant over the course of the year. Many students will compute the average of the output values for 2011 and 2012 and use this number as the proposed output value for March 1, 2012. Some students will compute the average rate of change for the population over the course of the year and use this number in their reasoning. Make sure that groups are carefully describing whatever method they are using, including how they are taking into account the fact that March 1 is not the midpoint of the year in question. There are no "right" or "wrong" strategies for approaching this problem, as long as students clearly articulate their methods for approximating the population of Rutherford County on March 1st.

c. Would the same strategy you used in Part (b) also allow you to estimate the population of Rutherford County on December 1, 2014? Would you need to adjust your strategy? Explain.

This problem is designed to make students think carefully about the details of their strategies. Students should complete this problem before the groups report out their strategies from Part (b).

d. Consider the strategies used by others in your class. On what assumptions is each strategy based? What strategy produces the most accurate approximation?
Have the students discuss the various strategies. Remember that the key observation is the assumption of the constant rate of change of the population with respect to the number of years elapsed since July 1, 2010. If students do not notice this, then you will have to bring it to their attention.

4. Iodine-132 is a radioisotope that is commonly used in medical procedures. A technician injects 100 micrograms of Iodine-132 into a patient undergoing thyroid therapy and measures the amount of the substance present every 24 hours for five days. Let $n$ represent the number of hours elapsed since the technician injected the 100 micrograms of Iodine-132 into the patient, and let $f(n)$ represent the amount (in micrograms) of Iodine-132 present in the patient $n$ hours after the treatment was administered.

| $n$ | 24 | 48 | 72 | 96 | 120 |
|---|---|---|---|---|---|
| $f(n)$ | 28.358 | 8.042 | 2.280 | 0.647 | 0.052 |

a. Does the amount of Iodine-132 present in the patient vary at a constant rate with respect to the number of hours elapsed since the technician injected the initial 100 micrograms of Iodine-132 into the patient? Explain.
The amount of Iodine-132 present in the patient does not vary at a constant rate with respect to the number of hours elapsed since the technician injected the initial 100 micrograms because changes in the amount of Iodine-132 present in the patient are not proportionally related to changes in the number of hours elapsed since the technician administered the treatment. For example, we notice that the change in the amount of Iodine-132 present in the patient during the first 24 hours after the treatment was administered (–71.642) is not equal to the change in the amount of Iodine-132 present in the patient during the second 24 hour interval after the treatment was administered (–20.316). The changes in the amount of Iodine-132 present in the patient are therefore not related by a constant multiple to the changes in the number of hours elapsed since the technician administered the initial treatment of 100 micrograms, which implies that these quantities do not vary at a constant rate with respect to each other.

b. Approximate the amount (in micrograms) of Iodine-132 present in the patient 56 hours after the technician administered the initial treatment of 100 micrograms of Iodine-132. Explain how you computed your approximation.
There are many ways to answer this problem. Encourage students to be thoughtful about their approaches and to identify the assumptions on which these approaches are based. Students will likely assume, even if not explicitly, that the account value varies at a constant rate with respect to the number of months elapsed since the initial deposit. It is important that you make students aware of these assumptions before moving to the next problem.
Let's begin by assuming a constant rate of change of $f(n)$ with respect to $n$ from $n = 48$ to $n = 72$. This assumed constant rate of change is given by the average rate of change of the function $f$ over this interval:

$$\frac{f(72)-f(48)}{72-48}=\frac{-5.762}{24}=-0.240$$

The change in the amount of Iodine-132 present in the patient from $n = 48$ to $n = 56$ is approximately the product of this change in the amount of time elapsed since the technician administered the treatment and average rate of change of $f$ over the interval [48, 72]. The value we're approximating, $f(56)$, is given by the sum of $f(48)$ and the change in the amount of Iodine-132 present in the patient from $n = 48$ to $n = 56$. Therefore, the amount of Iodine-132 (in micrograms) present in the patient 56 hours after the technician administered the initial treatment is given by $8.042 - 0.240(56 - 48) = 6.122$.

c. Approximate how much less Iodine-132 is present in the patient 27 hours after the technician administered the treatment then there was 21 hours after the technician administered the treatment. Explain how you computed your approximation.

As with Part (b), encourage students to be thoughtful about their approaches and to identify the assumptions on which these approaches are based.

The interval of time between 21 and 27 hours after the technician administered the treatment of Iodine-132 is 6 hours. If we assume a constant rate of change of the amount of Iodine-132 present in the patient and the number of hours elapsed since the technician administered the treatment, over the interval from $n = 24$ to $n = 48$, then the change in the number of micrograms of Iodine-132 present in the patient from $n = 21$ to $n = 27$ is given by the product of the assumed constant rate of change and 6, the change in time between 21 and 27 hours after the technician administered the treatment. That is, the change in the amount of Iodine-132 present in the patient (in micrograms) from $n = 21$ to $n = 27$ is given by

$$\left(\frac{f(48)-f(24)}{48-24}\right)\cdot 6 = \left(\frac{8.042-28.358}{24}\right)\cdot 6 = -5.079.$$

When examining function behavior, it is often very useful to assume that the output quantity varies at a constant rate with respect to the input quantity over particular intervals of the domain, even when we know it does not. In other words, it is often very useful to assume a ***local constant rate of change*** of the output quantity with respect to the input quantity. We use the phrase, "local constant rate of change" because we do *not* assume that the output quantity varies at a constant rate with respect to the input quantity over the entire domain, but only over a part of the domain.

The reason why it is useful to assume a local constant rate of change is because then the changes in the quantity's measures are proportional. In Problems 4 and 4, you assumed that the output quantity varies at a constant rate with respect to the input quantity, and used the proportionality in the changes in the quantity's measures to approximate the output values associated with particular input values. As you will see in the next investigation, assuming a local constant rate of change is useful for determining the rate at which the output quantity changes with respect to the input quantity at a specific value of the input quantity.

**Homework**

1. Suppose you deposit $1,500 into an account at the Fine Bank of Murfreesboro on August 1, because the account value increases continuously for all student accounts. You are very disciplined and don't withdraw any money from your account for five months. The following is a table of values showing the amount of money in your bank account at the end of each of those five months.

| **Date** | August 31 | September 30 | October 31 | November 30 | December 31 |
|---|---|---|---|---|---|
| **Account Value** | $1,502.73 | $1,505.46 | $1,508.20 | $1,510.94 | $1,513.69 |

a. Does the account value vary at a constant rate with respect to the number of months elapsed since August 1st? Explain.

The account value does not vary at a constant rate with respect to the number of months elapsed since August 1st because changes in the account value are not proportionally related to changes in the number of months elapsed since the initial deposit. For example, we notice that the increase in the account value during the first month after the initial deposit ($2.73) is not equal to the increase in the account value during the second month after the initial deposit ($2.74). The changes in the account value are therefore not related by a constant multiple to the changes in the number of months elapsed since the initial deposit, which implies that these quantities do not vary at a constant rate with respect to each other.

b. Approximate how much money is in your account on October 7th. Explain how you computed your approximation.

There are many ways to answer this problem. Students will likely assume, even if not explicitly, that the account value varies at a constant rate with respect to the number of months elapsed since the initial deposit.

We notice that October 7th is one week (¼th of a month) after September 30th. Therefore, the change in the account value during this one-week period of time is approximately ¼th of the change in the account value from September 30th to October 31st. Therefore, the approximate account value (in dollars) on October 7th is given by 1505.46 + ¼(1508.20 – 1505.46) = 1506.14. This solution is based on the assumption of a constant rate of change of the account value with respect to the number of months elapsed since the initial deposit over the interval of time from September 30th to October 31st.

c. Approximate how much more money there is in your account on November 13th than there is on November 5th. Explain how you computed your approximation.

The interval of time between November 5th and November 13th is 8 days, or $^{8}/_{30}$ of a month. If we assume a constant rate of change of the account value with respect to the number of months elapsed since the initial deposit over the interval of time from October 31st to November 30th, then the change in the account value from November 5th to November 13th is given by the product of the assumed constant rate of change and the change in months between November 5th and November 13th. That is, the change in the account value (in dollars) from November 5th to November 13th is given by ($^{8}/_{30}$)(1510.94 – 1508.20) ≈ 0.73.

2. A car's distance past a stop sign increases according to the function $s$ defined by $s(t) = 2t^2$ where $t$ represents the number of seconds elapsed since the car started moving and $s(t)$ represents the car's distance (in feet) past the stop sign.

a. What is the average rate of change of the car's distance past the stop sign as the value of $t$ increases from $t = 3$ to $t = 10$.

b. What does the average rate of change you computed in Part (a) represent in the context of this problem?

c. Estimate how fast the car is moving when $t = 3$ by computing the average rate of change of the car's distance past the stop sign $s(t)$ as $t$ changes from

i. 2.9 to 3 ii. 2.95 to 3 iii. 2.98 to 3 iv. 3 to 3.02 v. 3 to 3.05

i. $\dfrac{2(3)^2 - 2(2.9)^2}{3 - 2.9} = 11.8$

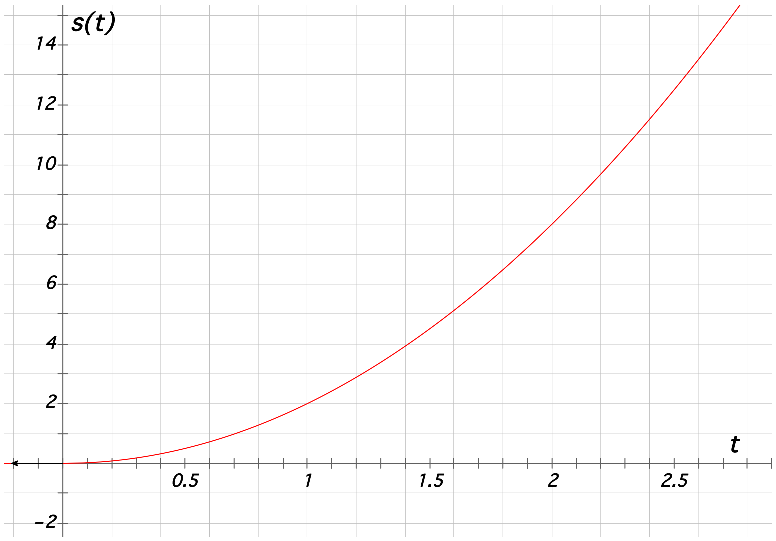
ii. $\dfrac{2(3)^2 - 2(2.95)^2}{3 - 2.95} = 11.9$

iii. $\dfrac{2(3)^2 - 2(2.98)^2}{3 - 2.98} = 11.96$

iv. $\dfrac{2(3.02)^2 - 2(3)^2}{3.02 - 3} = 12.04$

v. $\dfrac{2(3.05)^2 - 2(3)^2}{3.05 - 3} = 12.1$

d. Construct a graph of $s$ on the axes below and describe how the car's average rate of change changes from $t = 2.9$ to $t = 3.1$.



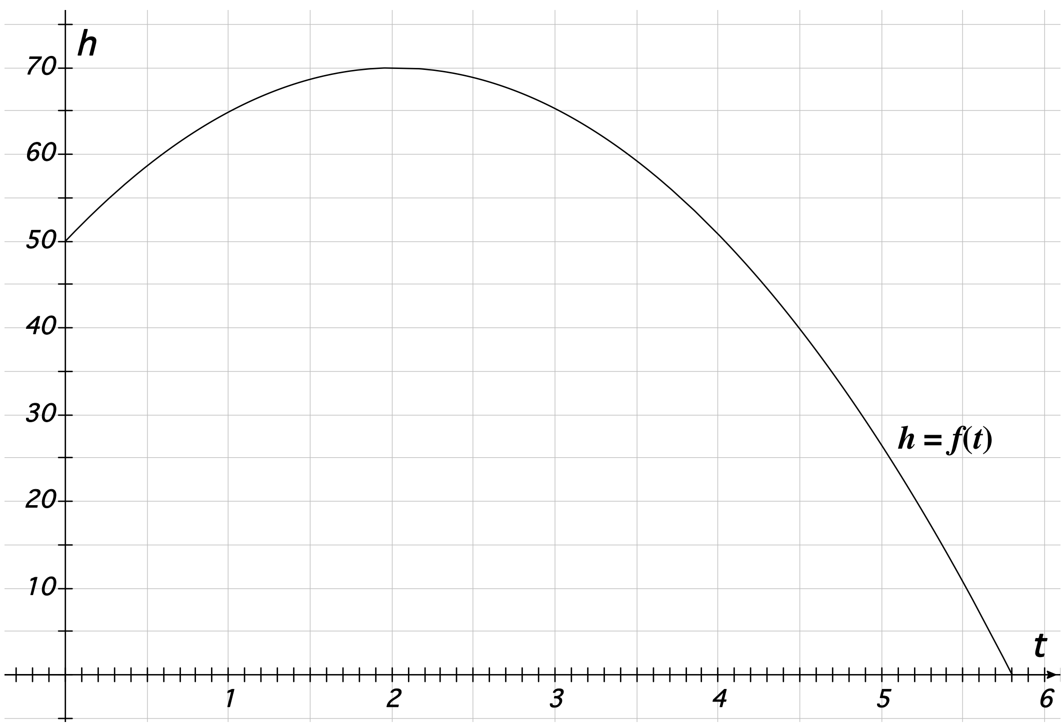
This investigation asks students to approximate the local constant rate of change of one quantity with respect to another at a specific value. We support students' informal and intuitive understanding of a limiting process by asking them to compute average rates of change over increasingly smaller intervals. We also support students' understanding of the slope of the line tangent to the graph of a function at a specific point as a geometric interpretation of the limiting value of average rates of change. Rather than providing a graph or a table of values for students to use to approximate the local constant rate of change of one quantity with respect to another, Problems 5-9 provide a function definition $f$. Students then examine a formula (symbolically and graphically) for the average rate of change of $f$ over the interval $[a, a + h]$ for some fixed $a$ and variable $h$. In doing so, students develop a more robust understanding of the limiting process, and begin to see $h$, the length of the interval over which the average rate of change is computed, as a varying quantity.

1. Over the holidays, you and your friends drove from Phoenix to Flagstaff for a ski trip. While pulling out of your driveway you noticed that your car's speedometer was broken. Since you had received a speeding ticket the prior week, it was important that you kept track of your speed to avoid receiving another ticket.
   Consider discussing Parts (a) and (b) as a class. We encourage you to let students discuss Part (c) in their groups or with another person in the class for a few minutes before calling on students to report their thinking.
   a. What quantities could you use to ***estimate*** your speed as you pass by the Montezuma Castle exit (a popular speed trap) on your drive to Flagstaff?
      A common response is that students say they would estimate their distance and time without specifying what distance and what time. Ask them if they are talking about the distance of the car behind a truck, the distance of the car from Flagstaff, or even something more absurd like the distance of your car from Kansas City. Similarly ask students to specify what amount of time they might measure.
   b. What units could you use to measure each of these quantities?
      Students should recognize that since speed is not directly measurable (without the speedometer), students will need to determine the speed indirectly by determining how the values of the quantities *distance from some position or point* and *elapsed time since passing some position or point* change together. Prompt students to provide a specific description of the quantities they identify. For example, if a student identifies "distance" as a quantity, ask him/her to describe the distance the student intends to measure (e.g., "I'm measuring the distance the car has traveled since leaving Phoenix…"). When defining a variable to represent the values that this distance can take on, one must include the unit of measurement (e.g., number of miles); otherwise it will not be clear what statements like "the distance the car has traveled since leaving Phoenix is 10" mean. The same detailed description should be required for the quantity "time".
   c. Choose two specific values for each of the quantities described in Parts (a) and (b) and explain how to use these values to estimate the speed of the car.
      *Answers may vary*. Suppose we travel 1.2 miles in 90 seconds (1.5 minutes). Then
      - We would travel 2.4 miles in 3 minutes (doubling the time elapsed and distance traveled) if we maintained the same speed.
      - We would travel 24 miles in 30 minutes if we maintained the same speed.
      - We would travel 48 miles in 60 minutes (1 hour) if we maintained the same speed.
      - *We estimate our speed to be about 48 miles per hour.*

      Probe students to specify an amount of distance that is traveled by the car in some specific amount of time. If a student determined the amount of time to travel from one mile-marker to the next, he might say that he traveled 1 mile in 1 minute and 15 seconds or 1 mile per 1.25 minutes or 48 miles per hour. This task provides students the opportunity to describe the method they used in Parts (a) through (c) to determine the car's speed over smaller and smaller intervals of time. As students present their methods, prompt them to describe the meaning of speed over some interval of time with the goal of supporting the development of the idea of average speed as the constant speed to cover a specific distance in some specified amount of time. Allow students to rely on their informal intuitions to compute numerical values of average speed and to say what these values represent.

d. Is it possible to determine your exact speed as you pass by Montezuma's Castle? Does your speedometer report your instantaneous speed? Explain.
*Answers may vary. Sample answers are provided.*
It is not possible to determine our exact speed. We can only estimate the speed by assuming a constant speed over some interval of time and distance that we are able to record and use the size of these intervals to determine that speed. Our speedometer also does not report exact speeds. It uses very small intervals of time elapsed and distance traveled to determine speeds (based on how many rotations the tires make in some amount of time).
This question can lead to some rich discussions, with many students arguing that the exact speed is the number that the needle points to on their speedometer, or the digits that are displayed on the dashboard. Guide the discussion so that students emerge with the understanding that in the physical world speed is determined by examining an amount of change of one quantity relative to an amount of change of another quantity. The better the measurement devices the more accurate of measurement is possible because one can measure changes in the two quantities over smaller intervals.

2. Toby is standing on top of a five-story building. He leans over the edge of the building and tosses a penny vertically into the air, then steps back and watches it rise and then fall to the ground. The following graph of the function $f$ shows the relationship between the height $h$ in feet of the penny above the ground and the time $t$ in seconds since Toby tossed the penny.



a. Does the graph above show the *actual* path that the penny took? Explain your answer.
This question should be addressed *briefly* (no more than five minutes) in small groups, and the instructor should call on several groups for answers. Most students will understand that the graph does not represent the actual path of the penny, because the problem says that Toby "threw the penny vertically." This does not, however, mean that students understand what the graph represents. Typically, some (up to half the class at times) will believe the graph shown actually represents the path of the penny. Students frequently say that "there is not enough information given to know exactly what path the penny took," and use this as justification that the graph *could* represent the penny's path. Regardless of the responses given, the instructor should direct students' attention to the information the graph provides. Consider asking questions like, "What are the input and output quantities for the function?" and "Can you explain how the quantities are varying together?" Demanding specific answers to these questions usually clears up inaccurate conceptions about the relationship between the graph and the path of the penny.

b. Use the table below to determine the average velocity of the penny on the given time intervals.

| ***t*** | 4.00 | 3.40 | 3.20 | 3.10 | 3.00 |
|---|---|---|---|---|---|
| ***f(t)*** | 50.80 | 60.68 | 63.18 | 64.29 | 65.30 |

i. Time interval $3.00 \le t \le 4.00$.
The average rate of change of the penny on this interval is

$$\frac{\Delta f(t)}{\Delta t} = \frac{50.80 - 65.30}{4 - 3} = -14.5 .$$

ii. Time interval $3.00 \le t \le 3.40$.
The average rate of change of the penny on this interval is

$$\frac{\Delta f(t)}{\Delta t} = \frac{60.68 - 65.30}{3.4 - 3} = -11.55 .$$

iii. Time interval $3.00 \le t \le 3.20$.
The average rate of change of the penny on this interval is

$$\frac{\Delta f(t)}{\Delta t} = \frac{63.18 - 65.30}{3.2 - 3} = -10.6 .$$

iv. Time interval $3.00 \le t \le 3.10$.
The average rate of change of the penny on this interval is

$$\frac{\Delta f(t)}{\Delta t} = \frac{64.29 - 65.30}{3.1 - 3} = -10.10 .$$

c. Use the table below to determine the average velocity of the penny on the given time intervals.

| ***t*** | 2.00 | 2.40 | 2.80 | 2.90 | 3.00 |
|---|---|---|---|---|---|
| ***f(t)*** | 70.00 | 69.30 | 67.02 | 66.21 | 65.30 |

i. Time interval $2.00 \le t \le 3.00$.
The average rate of change of the penny on this interval is

$$\frac{\Delta f(t)}{\Delta t} = \frac{65.30 - 70.00}{3 - 2} = -4.7 .$$

ii. Time interval $2.40 \le t \le 3.00$.
The average rate of change of the penny on this interval is

$$\frac{\Delta f(t)}{\Delta t} = \frac{65.30 - 69.30}{3 - 2.4} = -6.67 .$$

iii. Time interval $2.80 \le t \le 3.00$.
The average rate of change of the penny on this interval is

$$\frac{\Delta f(t)}{\Delta t} = \frac{65.30 - 67.02}{3 - 2.8} = -8.6\,.$$

iv. Time interval $2.90 \leq t \leq 3.00$.
The average rate of change of the penny on this interval is

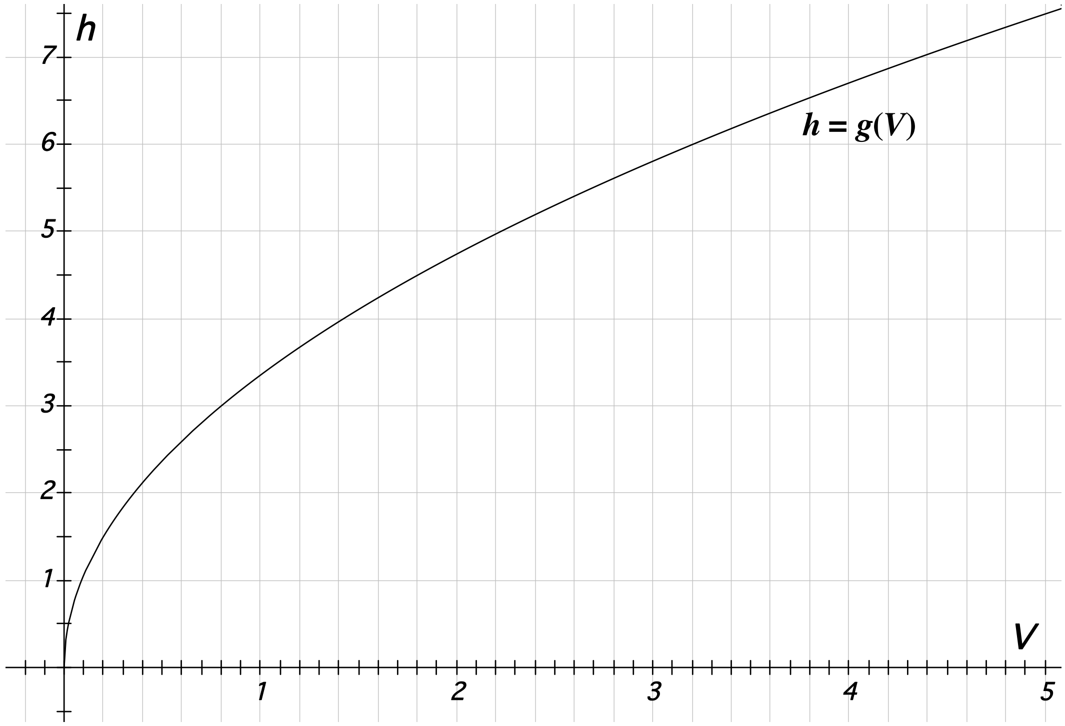
$$\frac{\Delta f(t)}{\Delta t} = \frac{65.30 - 66.21}{3 - 2.9} = -9.1\,.$$

d. Use the graph of $f$ to approximate the local constant rate of change of $f(t)$ with respect to $t$ around $t = 3$. Illustrate on the graph the value of your approximation.
Students' answers will vary. Do not be too concerned at the moment if students' responses are not very accurate. Subsequent tasks in this investigation will support students' understanding of how to determine a function's local constant rate of change. It is important, however, to make sure that students' estimates are based on a multiplicative comparison of a change in $t$ around $t = 3$ and the corresponding change in $f(t)$. Students should illustrate something like the slope of the line tangent to the graph of $f$ at $t = 3$ as a representation of the value of their approximation.

e. What is the meaning of the slope of the line tangent to the graph of $f$ at $t = 3$ in the context of this situation? What are the units of this slope?
Students should recognize that the slope of the line tangent to the graph of $f$ at $t = 3$ is essentially indistinguishable from the local constant rate of change of $f(t)$ with respect to $t$ over a very small interval including $t = 3$. Some important questions for students to consider while discussing this problem include: "How do we define velocity?" and "Can the slope of the tangent line represent the penny's velocity considering way we have defined (average) velocity?" Use this task to support students' recognition that the notion of instantaneous rate of change is a theoretical construct, or abstract idea.

f. How do the average velocities you computed in Parts (b) and (c) compare to the local constant rate of change you approximated in Part (d)?
Students should notice that as the interval over which they compute the average rate of change of $f(t)$ with respect to $t$ decreases, the value of average rate of change becomes closer to the local constant rate of change of $f(t)$ with respect to $t$ around $t = 3$.

3. Water is being poured into a bottle at a constant rate. The graph below represents the relationship between the height $h$ of water in the bottle (in centimeters) as a function of the volume $V$ of water in the bottle (in cubic centimeters). Let $h = g(V)$ denote the function represented by this graph.




a. The table below gives some approximate output values of the function $g$ for input values around $V = 1.6$. Use the table below to approximate the average rate of change of the height of water in the bottle (in centimeters) with respect to the volume of the water in the bottle (in cubic centimeters) on the given intervals of $V$.

| Value of $V$ | 1.40 | 1.45 | 1.50 | 1.55 | 1.60 |
|---|---|---|---|---|---|
| Approximate value of $g(V)$ | 3.964 | 4.034 | 4.103 | 4.171 | 4.237 |

i. Volume interval $1.40 \le V \le 1.60$.
The average rate of change of the height of water in the bottle (in centimeters) with respect to the volume of the water in the bottle (in cubic centimeters) is

$$\frac{\Delta g(V)}{\Delta V} = \frac{4.237 - 3.964}{1.60 - 1.40} = 1.365.$$

ii. Volume interval $1.45 \le V \le 1.60$.
The average rate of change of the height of water in the bottle (in centimeters) with respect to the volume of the water in the bottle (in cubic centimeters) is

$$\frac{\Delta g(V)}{\Delta V} = \frac{4.237 - 4.034}{1.60 - 1.45} = 1.353.$$

iii. Volume interval $1.50 \le V \le 1.60$.
The average rate of change of the height of water in the bottle (in centimeters) with respect to the volume of the water in the bottle (in cubic centimeters) is

$$\frac{\Delta g(V)}{\Delta V} = \frac{4.237 - 4.103}{1.60 - 1.50} = 1.34.$$

iv. Volume interval $1.55 \le V \le 1.60$.
The average rate of change of the height of water in the bottle (in centimeters) with respect to the volume of the water in the bottle (in cubic centimeters) is

$$\frac{\Delta g(V)}{\Delta V} = \frac{4.237 - 4.171}{1.60 - 1.55} = 1.32 .$$

b. The table below gives some approximate output values of the function $g$ for input values around $V = 1.6$. Use the table below to approximate the average rate of change of the height of water in the bottle (in centimeters) with respect to the volume of the water in the bottle (in cubic centimeters) on the given intervals of $V$.

| Value of $\boldsymbol{V}$ | 1.80 | 1.75 | 1.70 | 1.65 | 1.60 |
|---|---|---|---|---|---|
| Approximate value of $\boldsymbol{g(V)}$ | 4.495 | 4.432 | 4.368 | 4.303 | 4.237 |

i. Volume interval $1.60 \le V \le 1.80$.
The average rate of change of the height of water in the bottle (in centimeters) with respect to the volume of the water in the bottle (in cubic centimeters) is

$$\frac{\Delta g(V)}{\Delta V} = \frac{4.495 - 4.237}{1.80 - 1.60} = 1.29 .$$

ii. Volume interval $1.60 \le V \le 1.75$.
The average rate of change of the height of water in the bottle (in centimeters) with respect to the volume of the water in the bottle (in cubic centimeters) is

$$\frac{\Delta g(V)}{\Delta V} = \frac{4.432 - 4.237}{1.75 - 1.60} = 1.3 .$$

iii. Volume interval $1.60 \le V \le 1.70$.
The average rate of change of the height of water in the bottle (in centimeters) with respect to the volume of the water in the bottle (in cubic centimeters) is

$$\frac{\Delta g(V)}{\Delta V} = \frac{4.368 - 4.237}{1.70 - 1.60} = 1.31 .$$

iv. Volume interval $1.60 \le V \le 1.65$.
The average rate of change of the height of water in the bottle (in centimeters) with respect to the volume of the water in the bottle (in cubic centimeters) is

$$\frac{\Delta g(V)}{\Delta V} = \frac{4.303 - 4.237}{1.65 - 1.60} = 1.32 .$$

c. Use the graph above to approximate the local constant rate of change of $g(V)$ with respect to $V$ around $V = 1.6$. Illustrate on the graph the value of your approximation.
Students' answers will vary. Do not be too concerned at the moment if students' responses are not very accurate. Subsequent tasks in this investigation will support students' understanding of how to determine a function's local constant rate of change. It is important, however, to make sure that students' estimates are based on a multiplicative comparison of a change in $V$ around $V = 1.6$ and the corresponding change in $g(V)$. Students should illustrate something like the slope of the line tangent to the graph of $g$ at $V = 1.6$ as a representation of the value of their approximation.

d. What is the meaning of the slope of the line tangent to the graph of $g$ at $V = 1.6$ in the context of this situation? What are the units of this slope?
Students should recognize that the slope of the line tangent to the graph of $g$ at $V = 1.6$ is essentially indistinguishable from the local constant rate of change of $g(V)$ with respect to $V$ over a very small interval including $V = 1.6$.

e. How do the average velocities you computed in Parts (a) and (b) compare to the local constant rate of change you approximated in Part (c)?

Students should notice that as the interval over which they compute the average rate of change of $g(V)$ with respect to $V$ decreases, the value of average rate of change becomes closer to the local constant rate of change of $g(V)$ with respect to $V$ around $V = 1.6$.

4. Suppose that $y = f(x)$ is a function. Explain the meaning of the following statements:
   a. The average rate of change of $f(x)$ with respect to $x$ on the interval $a \le x \le b$.
   The average rate of change of a function $f$ from $x = a$ to $x = b$ is the constant rate of change of a linear function $g$ that has the same change in output as the function $f$ over the interval $[a, b]$.
   b. The instantaneous rate of change of $f(x)$ with respect to $x$ at $x = a$.
   The instantaneous rate of change for the function $f$ at the input value $x = a$ is the constant rate of change of a linear function $g$ with the following special properties:
   - $g(a) = f(a)$ (The graphs of the functions $f$ and $g$ intersect at $x = a$.)
   - The constant rate of change for the function $g$ can be approximated better and better by average rates of change for the function $f$ taken over smaller and smaller input intervals that begin or end at $x = a$.

5. The tables below provide the average rate of change of a function $y = f(x)$ on some input intervals that begin or end at $x = 2$.

| Interval of input values | $1.8 \le x \le 2$ | $1.9 \le x \le 2$ | $1.95 \le x \le 2$ | $1.99 \le x \le 2$ |
|---|---|---|---|---|
| Average rate of change of $f$ | 4.023 | 4.022 | 4.017 | 4.012 |

| Interval of input values | $2 \le x \le 2.01$ | $2 \le x \le 2.05$ | $2 \le x \le 2.10$ | $2 \le x \le 2.15$ |
|---|---|---|---|---|
| Average rate of change of $f$ | 5.010 | 5.013 | 5.024 | 5.145 |

Do you think the instantaneous rate of change of the function $f$ exists at the input value $x = 2$? Justify your response.

Students must look carefully at both tables before drawing any conclusions. Students should observe that average rates of change taken on smaller and smaller input intervals that begin at $x = 2$ appear to be approaching $y \approx 4.01$. On the other hand, average rates of change taken on smaller and smaller input intervals that end at $x = 2$ appear to be approaching $y \approx 5.01$. The information provided in this table is not conclusive, but it suggests that these two sequences of average rates of change are not approaching the same number as the input intervals get smaller and smaller. Since the sequences do not appear to be approaching the same number, it is likely (though not conclusive) that the instantaneous rate of change for the function $f$ is not definable at the input value $x = 2$. By the time students are working on this investigation, they have probably been exposed to limits and limit notation. If this is the case, some students might recognize that the information in the tables suggests that the left-hand and right-hand limits of the average rates of change probably exist but are not equal. Do not press for this kind of explanation; however, if it arises, ask students how they could make this notion more precise. For example, they have likely only worked with limits of *functions*. Some questions for these students to think about would include:

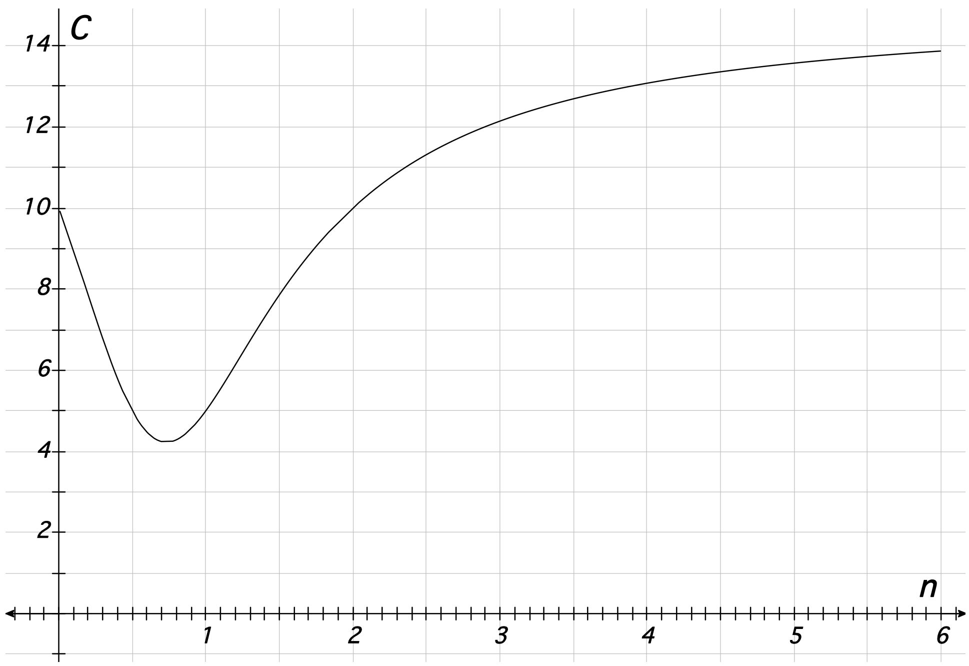
- Of what *function* are we considering the left-hand and right-hand limits?
- What input value are we approaching in the limit process?
- How would you use limit notation write your conclusions from the tables?

***Problems 6–10 refer to the following context:***

The international shipping conglomerate PEMDAS (Practically Everything Made Definitely Allows Shipping) is tracking the cost of painting its cargo ships. PEMDAS employs a team of painters but often needs to hire extra painters to get a ship painted quickly. The relationship between cost $C$ (in millions of dollars) for repainting a cargo ship and the number $n$ of extra painters (in hundreds) hired for the job is well approximated by the formula

$$C = f(n) = 15 - \frac{5(1+n)^2}{1+n^3}$$

when the number of hires is between 0 and 600. (Costs escalate rapidly when more than 600 extra painters are hired, and this formula is no longer valid.) The following is a graph of the function $f$ on its relevant domain.

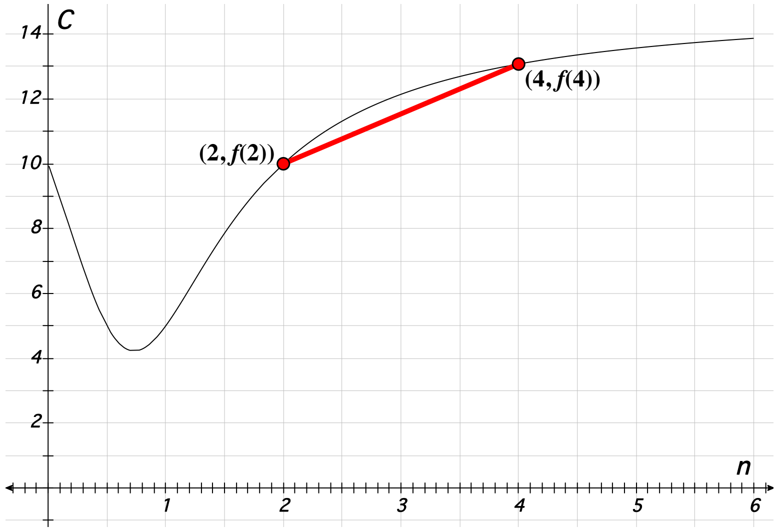



6. a. Using function notation, write an expression that computes the average rate of change in the cost of painting the ship with respect to hires on the interval $2 \le n \le 4$ hires.
Let $A$ represent the average rate of change of $f$ over the interval $2 \le n \le 4$ hires.

$$A = \frac{f(4) - f(2)}{4 - 2}$$

Students do not need to write out the algebraic formula for the value of $A$ in this problem, but it is okay if they want to do so. Students do not have to assign a letter name to the average rate of change, but if they do, make sure they do not use $C$ or $h$.

b. What does the average rate of change you expressed in Part (a) represent in the context of this situation? Represent this average rate of change on the graph of $f$.
The average rate of change represents the slope of the line segment connecting the points $(2, f(2))$ and $(4, f(4))$ on the graph of $f$ (see graph below). In the context of the problem, the average rate of change represents the constant rate that the cost of painting the ship would have to increase by in order to change from 10 million dollars to approximately 13 million dollars as the number of hires increases from 200 to 400.




c. Compute the average rate of change in the cost of painting the ship with respect to the number of hires on the interval $2 \le n \le 4$ hires.
The average rate of change in the cost of painting the ship with respect to the number of hires on the interval $2 \le n \le 4$ is approximately 1.5 million dollars per 100 hires.

7. Consider the function defined by $y = A(h)$ defined by the formula

$$y = A(h) = \frac{f(2+h) - f(2)}{(2+h) - 2}.$$

a. What are the units associated with the input variable $h$?
The input into the function $f$ must have "100's of hires" as its unit. Therefore, $h$ must also have this unit.

b. What are the units of the output variable $y$?
The units for $y$ must be "millions of dollars per 100 hires" (or "tens of thousands of dollars per hire").

c. What is the approximate value of $A(3)$?
$A(3) = \frac{f(2+3) - f(2)}{(2+3) - 2} = \frac{f(5) - f(2)}{3} = \frac{13.6 - 10}{3} = 1.2$.

d. What does the output of the function $A$ represent in the context of the painting problem?
The values of the function $A$ represent the average rate of change for the cost of painting the ship with respect to hundreds of hires on the input interval $2 + h \le n \le 2$ or $2 \le n \le 2 + h$.

e. What is the relevant domain for the input variable to the function $A$?
From an algebraic perspective, we know that $h$ cannot be 0. Based on the relevant domain for the function $f$ provided in the introduction to this investigation, the relevant domain for the function $A$ would be $-2 \le h < 0 \ \cup \ 0 < h \le 4$.

8. a. Use the following formulas for the functions $f$ and $A$ to fill in the tables below.

$$C = f(n) = 15 - \frac{5(1+n)^2}{1+n^3} \qquad y = A(h) = \frac{f(2+h) - f(2)}{(2+h) - 2}$$

| Value of $h$ | 1.00 | 0.75 | 0.20 | 0.10 | 0.01 |
|---|---|---|---|---|---|
| Value of $f(2+h)$ | 12.143 | 11.774 | 10.604 | 10.317 | 10.033 |
| Approximate value of $A(h)$ | 2.143 | 2.366 | 3.022 | 3.172 | 3.317 |

| Value of $h$ | –1.00 | –0.50 | –0.25 | –0.05 | –0.01 |
|---|---|---|---|---|---|
| Value of $f(2+h)$ | 5.000 | 7.057 | 9.054 | 9.829 | 9.967 |
| Approximate value of $A(h)$ | 5.000 | 4.286 | 3.784 | 3.418 | 3.350 |

b. As the value of $h$ gets closer to zero, what are some things you notice in the tables above?
The values of $A(h)$ in each table are getting closer to each other (the difference between consecutive values is getting smaller in magnitude) as the values of $h$ gets closer to 0. The values of $A(h)$ are becoming fairly good estimates for the local constant rate of change of $f$ around $n = 2$ as $h$ approaches 0 from the left and the right.

c. Use the tables above to approximate the local constant rate of change of $f(n)$ with respect to $n$ around $n = 2$.
Answers will vary, so make sure that students can explain how they obtained their answers. Some students will take the average of the values for $A(0.01)$ and $A(-0.01)$. Using this approach would give a proposed local constant rate of change $m = 3.333$ (accurate to three decimal places). If students use this method, consider asking, "Is 3.333 the value of $A(0)$?" and "If we use a smaller value of $h$, do you think the average of the outputs would change?" As groups are discussing this problem (most will stop when they arrive at a numerical answer), ask them to think about the *process* going on in the table. Would letting $h$ get closer and closer to 0 allow for better approximations to the local constant rate of change (i..e, slope of the tangent line)? Why can't we just let $h = 0$ in the formula for the function $A$?

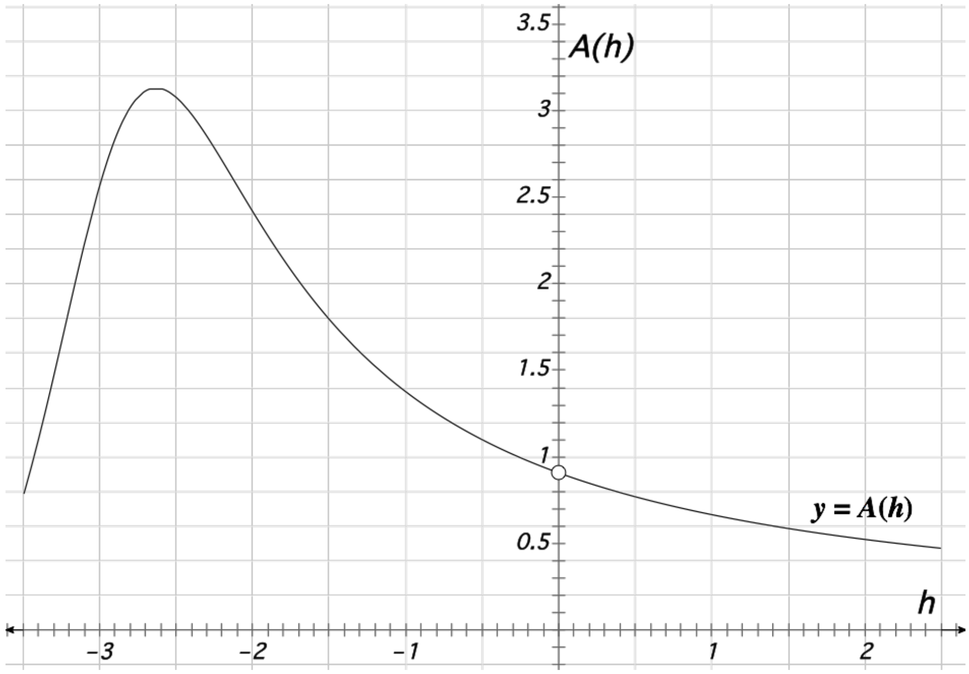
9. Consider the function $y = A(h)$ defined below.

$$y = A(h) = \frac{f(3.5+h) - f(3.5)}{(3.5+h) - 3.5}$$

a. What does the output of this function represent in the context of the ship-painting situation?
The outputs of the function $A$ represent the average rate of change in the cost of painting the ship with respect to the number of hires, as the number of hires increases from $n = 3.5 + h$ to $n = 3.5$ (if $h < 0$) or as the number of hires increases from $n = 3.5$ to $n = 3.5 + h$ (if $h > 0$).

b. What is the relevant domain for the function $A$?
The relevant domain for the function $A$ is the set $-3.5 \le h < 0 \;\cup\; 0 < h \le 2.5$.

c. The following is a graph of the function $A$ on its relevant domain.




The function $A$ has its maximum output (which is approximately 3.1) at input value $h \approx -2.63$. What does the point (–2.63, 3.1) represent in the context of the ship-painting situation?

This point tells us that the average rate of change in the cost of painting the ship with respect to the number of hires on input intervals that begin or end with $n = 3.5$ will be the largest on the interval $0.87 \le n \le 3.5$. From a geometric perspective, the secant line drawn from the point $(3.5, f(3.5))$ to the point $(3.5 + h, f(3.5 + h))$ will have the greatest positive slope when $h = -2.63$. Encourage those who use secant lines in their answer also to reason from the perspective of average rate of change.

***Note on Part (d):***

Reasoning with secant lines is not the most productive way to understand the process behind computing derivatives, so the "geometric perspective" on average rate of change is not emphasized in these investigations. Nonetheless, this perspective is widely used in calculus texts. The following problem provides an opportunity for students to remember that average rate of change on an input interval represents the *slope*, or constant rate of change, of a secant line. Problem 9 is optional.

d. The function $A$ has no negative output on its relevant domain. Why is this the case?

This is a problem that can best be answered from the geometric perspective of secant lines. Remind students that the average rate of change on an input interval has a geometric representation as the slope of a secant line. Students should draw (or imagine) secant lines to the graph of the function $f$ on input intervals that begin or end at $n = 3.5$. They can see that all such secant lines will have positive slope. If time permits, it is worthwhile to ask students if this is the case for *other* average rate of change functions that can be defined for the function $f$. For example, the average rate of change function

$$y = A(h) = \frac{f(0.5+h) - f(0.5)}{(0.5+h) - 0.5}$$

does have a negative output.

e. Why is there a hole in the graph of the function $A$ at $h = 0$?

There is a removable discontinuity of the graph of $A$ at $h = 0$ because function $A$ is not defined when $h = 0$.

f. What is the approximate value of the $y$-coordinate of the hole on the graph of $A$? What is the significance of this value?
The hole in the graph is approximately located at the point (0, 0.90). The $y$-coordinate of the hole in the graph of $A$ represents the local constant rate of change of $f(n)$ with respect to $n$ around the point $(3.5, f(3.5))$.

g. Use your graphing calculator to construct a table of output values for the function $A$ as the values of the input $h$ get very close to 0. (Try setting $h = -0.01$ for example, with an increment $\Delta h = 0.001$.) What do you notice?
Students should notice that the calculator displays "ERROR" for the output of $A$ when $h = 0$. Students should also notice that the output values seem to be "settling down" to a number between 0.907 and 0.908. Suggest that students try initializing the value of $h$ closer to zero (say $h = -0.001$) to better identify the number to which the output values of $A$ are "settling down."

10. a. Based on your thinking in this investigation, explain what the following expression represents. (Recall that the function $f$ is the cost function of painting the ship.)

$$\lim_{h\to 0}\frac{f(3.5+h)-f(3.5)}{(3.5+h)-3.5}.$$

Students are likely to struggle with this question, even if they have seen "limit notation" before. Do not accept a simple translation of the expression into words. We are interested in what the expression *means*, not just what the expression *says*! Make sure students attend to what each part of the expression represents. Ask students to describe what the following expressions represent:

- $f(3.5 + h) - f(3.5)$
- $(3.5 + h) - 3.5$
- $\dfrac{f(3.5+h)-f(3.5)}{(3.5+h)-3.5}$

Also ask students to describe what the notation $\lim_{h\to 0}[\square]$ means. After students have spent some time working on a statement in groups, ask a representative from one group to write its statement on the board. Let the class "wordsmith" the statement until everyone is satisfied with it. The final statement should read something like, "The expression represents the limiting value of the average rates of change of $f(n)$ with respect to $n$ on the interval $[3.5, 3.5 + h]$."

b. Consider the function $A$ defined by

$$y = A(h) = \frac{f(0.5+h)-f(0.5)}{(0.5+h)-0.5}.$$

i. Determine the relevant domain for the function $A$ in the context of the ship-painting situation.
$-0.5 \le h < 0 \;\;\bigcup\;\; 0 < h \le 5.5.$

ii. Is it appropriate to write the equation

$$A(0) = \lim_{h\to 0}\frac{f(3.5+h)-f(3.5)}{(3.5+h)-3.5}?$$

Justify your response.

While it may be tempting to understand the limit process above in this way, it is important to remember that $h = 0$ is not in the domain of the function $A$. Therefore, $A(0)$ is undefined.

iii. Use the table feature on your graphing calculator to estimate the value of

$$\lim_{h \to 0} \frac{f(3.5+h) - f(3.5)}{(3.5+h) - 3.5}.$$

How close to zero must the value of $h$ be for your estimate to be accurate to at least two decimal places? Explain how you determined your answer.

$$\lim_{h \to 0} \frac{f(0.5+h) - f(0.5)}{(0.5+h) - 0.5} \approx -6.66$$

This estimate was obtained by initializing the table at $h = -0.001$ and taking $\Delta h = 0.0001$. We know that this estimate is accurate to two decimal places because the output from the function $A$ is stable to two decimal places once $|h| < 0.0002$. (One can also argue that rounding up should be done to make the two-decimal-place estimate of the limit approximately –6.67; indeed, based on the table, the actual value of the limit appears to be a repeating decimal.)

**Homework**

1. Consider the cost-versus-number of hires function $C = f(n)$ explored in Problems 6–10 of this investigation.
   a. Using function notation, construct the formula for the function $y = A(h)$ whose output value for an input value of $h$ is the average rate of change for the function $f$ on the input interval $3 + h \le n \le 3$ (if $h$ is negative) or $3 \le n \le 3 + h$ (if $h$ is positive).
   $y = A(h) = \dfrac{f(3+h) - f(3)}{(3+h) - 3}$.
   b. Based on the context of the ship-painting situation, what is the relevant domain for the function $A$? $-3 \le h < 0 \ \cup \ 0 < h \le 3$.
   c. Use the following formulas for the functions $f$ and $A$ to fill in the tables below.

$$C = f(n) = 15 - \frac{5(1+n)^2}{1+n^3} \qquad y = A(h) = \frac{f(2+h) - f(2)}{(2+h) - 2}$$

| Value of ***h*** | 1.00 | 0.80 | 0.15 | 0.05 | 0.01 |
|---|---|---|---|---|---|
| Value of ***f*(3 + *h*)** | 13.077 | 12.938 | 12.330 | 12.208 | 12.156 |
| Approximate value of ***A*(*h*)** | 0.9341 | 0.9941 | 1.2498 | 1.3000 | 1.3211 |

| Value of ***h*** | –1.00 | –0.60 | –0.30 | –0.10 | –0.01 |
|---|---|---|---|---|---|
| Value of ***f*(3 + *h*)** | 10.000 | 11.101 | 11.691 | 12.005 | 12.130 |
| Approximate value of ***A*(*h*)** | 2.1429 | 1.7366 | 1.5078 | 1.3825 | 1.3319 |

   d. Use the tables above to approximate the local constant rate of change of $f(n)$ with respect to $n$ around $n = 2$.
   Student answers will vary; however, defensible answers would include

- "A number between 1.3319 and 1.3211."
- "1.3 accurate to one decimal place."
- "The average of 1.3211 and 1.3319."
- "About 1.33."
- "About 1.32."

There is not sufficient accuracy provided in these tables to draw a definitive conclusion. Suggest that students try taking smaller values of $h$ to discern greater accuracy. Good questions to ask include, "How could you tell from your data when the estimate is accurate to two decimal places?" and "How small in magnitude must the value of $h$ be to guarantee the estimate from both tables is accurate to three decimal places?"

2. Consider the function $a = f(b) = b^2$.

   a. What is the meaning of the function $A$ defined below?

$$y = A(x) = \frac{f(4+x) - f(4)}{(4+x)-4}.$$

   The output value of the function $y = A(x)$ for an input value of $x$ is the average rate of change of the function $f$ on the input interval $4 + x \le b \le 4$ (if $x$ is negative) or $4 \le b \le 4 + x$ (if $x$ is positive).

   b. Which of the following formulas is algebraically equivalent to the formula that defines the function $A$ in Part (a)?

   i. $y = \dfrac{16 + x^2 - 16}{x}$

   ii. $y = \dfrac{16 + 8x + x^2 - 16}{x}$

   iii. $y = \dfrac{x^2 + 8x}{x}$

   Formulas (ii) and (iii) are algebraically equivalent to the formula given in Part (a). Formula (ii) is obtained from the formula in Part (a) by simplifying the denominator and expanding the binomial in the numerator. Formula (iii) is obtained from Formula (ii) by simplifying the numerator.

   c. Explain how you could use the table feature on your graphing calculator and one of the equivalent formulas for the function $A$ to estimate the local constant rate of change of $f(n)$ with respect to $n$ around $n = 4$.

   In words, the procedure would be to enter one of the formulas, say Formula (iii) from Part (b), into the graphing calculator, initiate the table with some value close in magnitude to 0, and choose a table increment that allows the student to see the output of the average value function as values of the input variable approach 0. It is important to check outputs for input values that approach 0 from the left and from the right.

   d. Approximate the local constant rate of change of $f(n)$ with respect to $n$ around $n = 4$. How accurate do you think your approximation is? Explain your thinking.

   Students should be able to see that the average rates of change of $f(n)$ with respect to $n$ around $n = 4$ approaches 8.000 as $h$, the values of the input variable of $A$, approach 0 from the left and from the right. The average rates of change "step down" toward the value $m = 8.000$ in nearly linear fashion as the values of $h$ approach 0 from the right and "step up" toward the value $m = 8.000$ in nearly linear fashion as the values of $h$ approach 0 from the left. This behavior is especially obvious if students initiate their tables with $|x| \le 0.01$.

3. Consider the function $f$ defined by $f(x) = (x - 1)^{2/3} + 2$.
   a. Construct a graph of the function $f$ using your graphing calculator. Is the function $f$ locally linear at the input value $x = 1$? Explain.
   The function $f$ has a "vertical tangent" line at the input value $x = 1$ and therefore is not locally linear at this input value. Remember that a function is locally linear at an input value $x = a$ if the function has essentially a local constant rate of change over very small intervals around $x = a$. Because $f$ has a vertical tangent line at the input $x = 1$, $f$ has no local constant rate of change around $x = 1$, and is therefore not locally linear at this input value.
   b. Use the table feature on your graphing calculator to estimate the value of

$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{(1+h) - 1}.$$

   What do you notice as the value of $h$ approaches zero?
   As the value of $h$ approaches zero, the average rate of change of $f$ over the interval $[1, 1 + h]$ grows without bound.
   c. Use the table feature on your graphing calculator to estimate the value of

$$\lim_{h \to 0} \frac{f(3+h) - f(3)}{(3+h) - 3}.$$

   Explain what your solution represents. How close to 0 must the value of $h$ be in order to guarantee your estimate is accurate to at least three decimal places?

$$\lim_{h \to 0} \frac{f(3+h) - f(3)}{(3+h) - 3} \approx 0.529$$

   The value 0.529 is the local constant rate of change of $f(x)$ with respect to $x$ over a very small interval around $x = 3$. Requiring $|h| < 0.001$ is sufficient to guarantee three-decimal-place accuracy.

4. Consider the functions

$$y = f(h) = \frac{(3+h)^2 - 9}{(3+h) - 3} \qquad y = g(h) = \frac{h^2 + 6h}{h} \qquad y = j(h) = 6 + h$$

   a. What is the implied domain for each of these functions?
   The functions $f$ and $g$ are defined for all input values that are real numbers other than $h = 0$. The function $j$ is defined for all input values that are real numbers.
   b. Are each of these functions equivalent? Explain.
   Since $j(0)$ exists while $f(0)$ and $g(0)$ do not, it is clear that the function $j$ is not equal to the functions $f$ and $g$. On the other hand, if we sketch all three functions on a graphing calculator, we see only one graph. Consequently, it would appear that these functions have equal output for every other input value in their respective domains. This makes sense because the formula for the function $g$ can be obtained from the formula for function $f$ by expanding the binomial in the numerator of the formula for $f$ and simplifying the numerator and denominator. Likewise, the formula for the function $j$ can be obtained from the formula for the function $g$ by factoring out a common factor of $h$ from the numerator of the formula for $g$ and then cancelling.
   c. Is it appropriate to write the following string of equalities? Justify your response.

$$\lim_{h\to 0}\frac{(3+h)^2-9}{(3+h)-3}=\lim_{h\to 0}\frac{h^2+6h}{h}=\lim_{h\to 0}(6+h)=6$$

Yes, it is acceptable to write this string of equalities. We know that all three formulas produce the same output value for each non-zero input value. Therefore, it stands to reason that all three limiting processes should produce the same result.

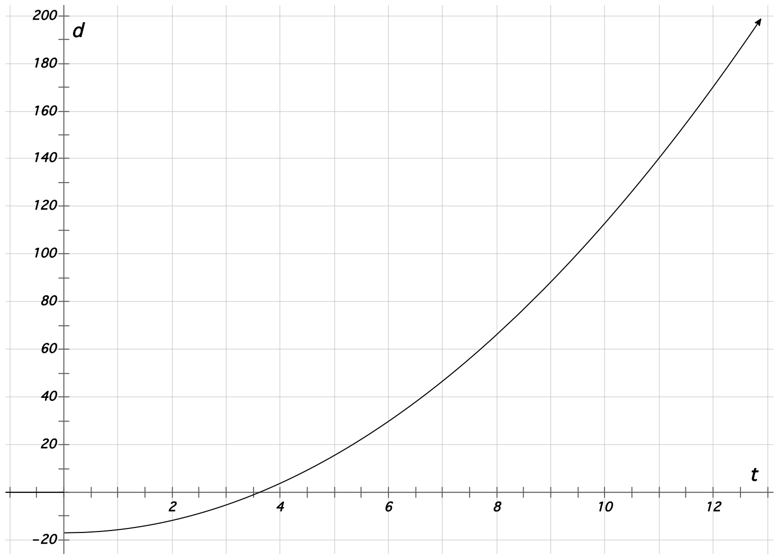
This investigation begins by prompting students to approximate the rate of change of one quantity with respect to another at a specific value of the input quantity. We expect students to leverage the meanings developed in previous investigations, namely the ideas of average and constant rate of change. Students should approximate "instantaneous" rates of change requested in Problems 1-3 by computing average rates of change over increasingly small intervals of the input quantity. These first three problems lay the foundation for the concept of local linearity as well as for the geometric and analytical interpretation of the derivative of a function at a point. Since Problems 1-3 are lengthy, feel free to ask students to complete Problem 1 or 2, or both, independently before class. You can then begin class by asking students to discuss their work with a partner or in small groups before proceeding with the remainder of the investigation.

Research has shown that many calculus students' understanding of derivatives is not grounded in robust meanings of rate of change. When prompted to explain what the derivative at a point represents, students often say things like, "The slope of the tangent line." This investigation supports students' understanding of derivative at a point as the limiting value of average rates of change (i.e., the limiting value of a multiplicative comparison of the changes in quantities' values), not a property of a geometric object. We emphasize the idea that "instantaneous rate of change" is a theoretical construct; for a rate of change to exist, there must be changes in quantities' values to multiplicatively compare. As students work on this investigation, consistently hold them accountable for leveraging the meanings of constant and average rate of change developed in previous investigations.

Since this investigation addresses a variety of topics (local linearity, derivative of a function at a point, and differentiability), expect this investigation to extend over 3-5 class sessions.

The understandings promoted in this investigation were heavily informed by the content in Chapter 4 of *Calculus: Newton Meets Technology* by Patrick Thompson, Mark Ashbrook, Stacy Musgrave, and Fabio Milner (Thompson et al., 2015). Our design of this investigation was also informed by the CLEAR Calculus materials developed by Michael Oehrtman and Jason Martin (Oehrtman & Martin, 2010).

1. A car is driving away from a traffic light. The distance $d$ (in feet) of the car from the traffic light $t$ seconds since the car started moving is given by the formula $d = 1.3t^2 - 17$.
   a. Draw a graph of the relationship between the car's distance $d$ (in feet) from the traffic light and the time $t$ (in seconds) since the car started moving.
   Subsequent parts of this question ask students to illustrate particular quantities on their graph, so it's important that students draw their graphs large.

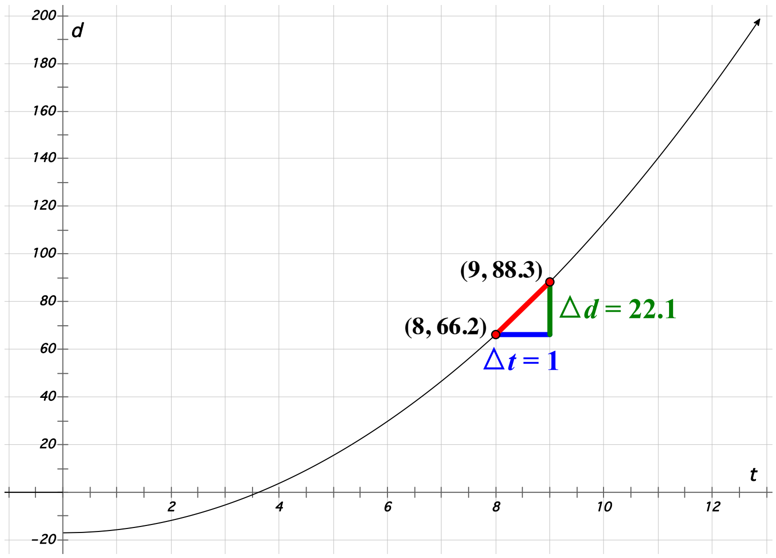



b. Does the car's distance (in feet) from the traffic light vary at a constant rate with respect to the number of seconds since the car started moving? Explain.
Students should say more than, "$d$ is not a linear function of $t$." Push students to explain why $d$ does not vary at a constant rate with respect to $t$ by attending to the multiplicative relationship between $\Delta d$ and $\Delta t$ for various values of $\Delta t$. In other words, students should use the function definition to show that $\Delta d$ and $\Delta t$ are not proportionally related. For example, suppose $t$ changes from $t = 0$ to $t = 2$. Then the corresponding change in $d$ is given by $\Delta d = 1.3(2)^2 - 17 - (1.3(0)^2 - 17) = 5.2$. Suppose $t$ changes from $t = 5$ to $t = 6$. Then the corresponding change in $d$ is given by $\Delta d = 1.3(6)^2 - 17 - (1.3(5)^2 - 17) = 14.3$. Since $\Delta d$ is not consistent for different changes in $t$, $\Delta d$ is not proportionally related to $\Delta t$. Therefore, the car's distance (in feet) from the traffic light does not vary at a constant rate with respect to the number of seconds elapsed since the car started moving. Students could also observe that for successive equal changes in the number of seconds elapsed since the car started moving, corresponding changes in the car's distance from the traffic light increases. Therefore, $\Delta d$ is not proportionally related to $\Delta t$.

c. Approximate the car's speed 8 seconds after it started moving and explain the strategy you used. Also, illustrate what your approximation represents on the graph you drew in Part (a) and explain how what you drew corresponds to the approximation value you computed.
Do not accept qualitative estimates as a response to this question. Encourage students to leverage the meanings developed in previous investigations. In particular, support them in recognizing the need to apply their understanding of average rate of change to approximate the car's speed 8 seconds after it started moving.
The car's average speed over the interval from $t = 8$ to $t = 9$ is an approximation of the car's speed 8 seconds after it started moving. This average rate of change is given by

$$\frac{1.3(9)^2 - 17 - (1.3(8)^2 - 17)}{9 - 8} = \frac{22.1}{1} = 22.1 \text{ ft/sec.}$$

To approximate the car's speed 8 seconds after it started moving, I computed the car's average rate of change over the interval $t = 8$ to $t = 9$. This average speed is the constant speed another car would have to travel to cover the same distance as the original car (22.1 ft) in the same one-second interval of time. This average rate of change is illustrated as the constant rate of change of the linear function that passes through the points (8, 66.2) and (9, 88.3) on the graph. This average rate of change is also the slope of the red line on the graph, or the number of times $\Delta d$ (represented by the vertical green line) is larger than $\Delta t$ (represented by the horizontal blue line).




d. Is your approximation from Part (c) an overestimate or underestimate? Justify your response by using language about the physical context, not the shape of the graph.
For successive equal changes in the number of seconds elapsed since the car started moving, the corresponding changes in the car's distance from the traffic light increase. The approximation from Part (c) is therefore an overestimate since it represents the average rate of change of the car's distance from 8 to 9 seconds after it started moving—a one-second interval of time immediately beyond the moment of time for which we're approximating the car's speed.

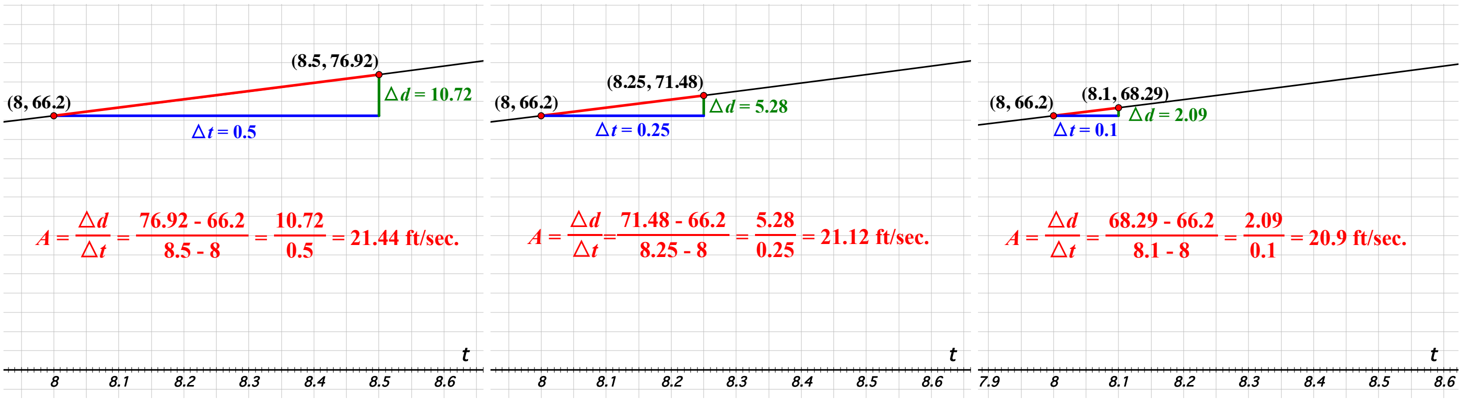
e. Define a variable to represent the value of the quantity you approximated in Part (c). Use this variable to represent the error of your approximation. (Note that the error of an approximation is the positive amount that the approximation differs from the value being approximated.)
Let $L$ represent the car's speed 8 seconds after it started moving. The error of the approximation from Part (c) is represented by either of the following expressions:

$$\left| L - \frac{1.3(9)^2 - 17 - (1.3(8)^2 - 17)}{9-8} \right| \quad \text{or} \quad \frac{1.3(9)^2 - 17 - (1.3(8)^2 - 17)}{9-8} - L.$$

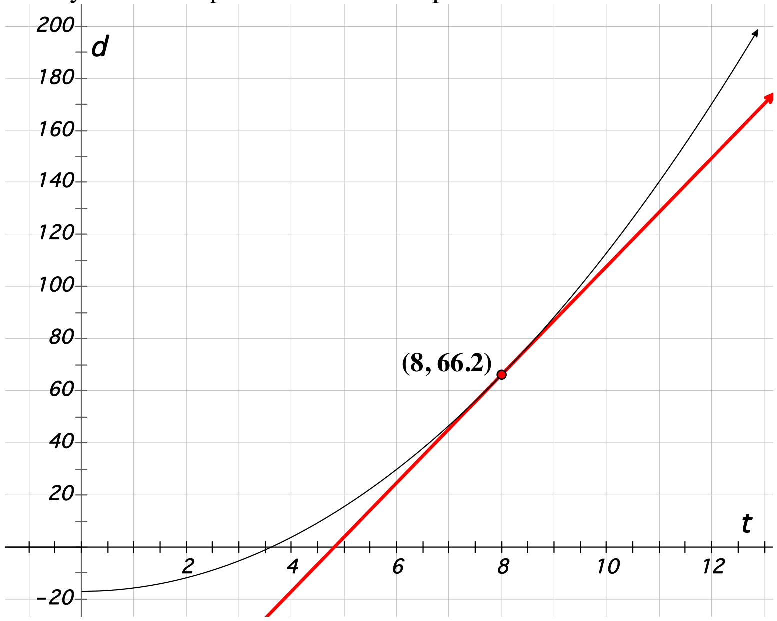
The first expression would accurately represent the error of the approximation even if it were an underestimate, and is for this reason more general than the second. The second expression is correct in this context only because the approximation from Part (c) is an overestimate of the car's speed 8 seconds after it started moving.

f. i. Explain how you might decrease the error of your approximation.
To decrease the error of the approximation, one could decrease the interval of time over which the car's average speed is computed. For example, computing the car's average speed over the interval from $t = 8$ to $t = 8.5$ would produce a better approximation than the one computed in Part (c).

ii. Compute three approximations that are more accurate than the one you calculated in Part (c) and represent these approximations on three "zoomed in" graphs of the function $d = 1.3t^2 - 17$.
Let $A$ represent the approximation of the car's speed 8 seconds since it started moving.




iii. Represent on the graph you drew in Part (a) the value you're approximating and explain how what you drew represents the car's speed 8 seconds after it started moving.




The constant rate of change, or slope, of the linear function tangent to the curve at $t = 8$ (i.e., that passes through the curve only at the point (8, 66.2)) represents the speed of the car 8 seconds since it started moving. The four approximations we've computed are average rates of change, which geometrically represent the constant rate of change, or slope, of linear functions that pass through two points on the curve, one of them being (8, 66.2). We've seen that as we decrease the interval over which we compute the car's average rate of change, the error of our approximation decreases, and the constant rate of change our approximations represent more closely resemble the constant rate of change of the linear function that is tangent to the curve at $t = 8$. As we make the interval over which we compute the car's average rate of change infinitely small, our average rate of change approximations become indistinguishable from the constant rate of change of the line tangent to the curve at $t = 8$. A geometric interpretation of the car's speed 8 seconds after it started moving is therefore the constant rate of change, or slope, of this tangent line.

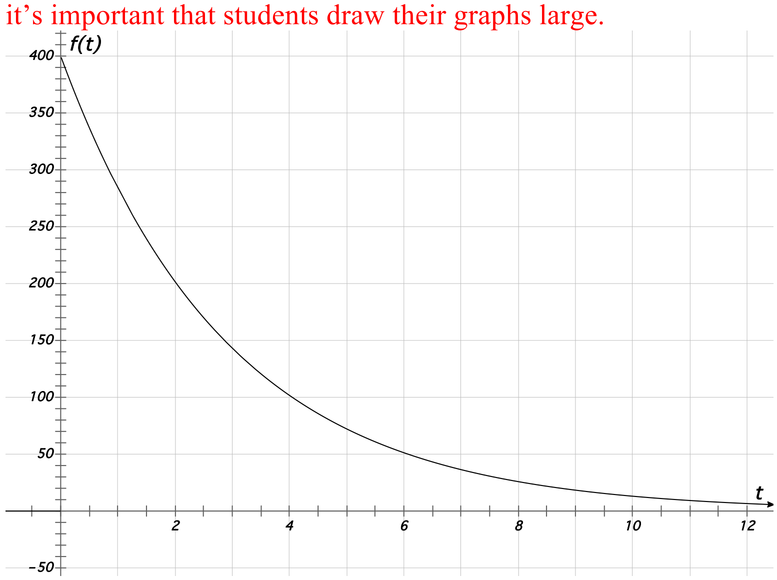
iv. Reflect on your responses to Part (ii) and (iii) to determine if there is any point at which it doesn't make sense to decrease the interval over which you're computing the average speed of the car to approximate the car's speed 8 seconds after it started moving. In other words, is there any point at which it doesn't make sense to decrease $\Delta t$ to get more accurate approximations? Explain. (Feel free to use graphs or equations to communicate your reasoning.)

For very small changes in time around $t = 8$, corresponding changes in the car's distance from the traffic light are essentially proportional. Therefore, over a really small interval of time around $t = 8$, the car's distance (in feet) from the traffic light ($d$) varies at essentially a constant rate with respect to the number of seconds elapsed since the car started moving ($t$). This constant rate of change is indistinguishable from the constant rate of change of the linear function tangent to the curve at $t = 8$, which represents the car's speed 8 seconds after it started moving. So, if we compute the car's average rate of change over an interval so small that the changes in the input and output quantities are essentially proportional (i.e., the output quantity varies at essentially a constant rate with respect to the input quantity), we gain very little by approximating the car's speed at $t = 8$ with an average rate of change over a smaller interval.

2. Vicki took 400 mg of Ibuprofen to relieve knee pain. The function $f(t) = 400(0.71)^t$ represents the amount of Ibuprofen in Vicki's body (in milligrams) in terms of the number of hours elapsed since Vicki took the initial dose of 400 mg.

   a. Draw a graph of the relationship between the amount of Ibuprofen in Vicki's body (in milligrams) and the number of hours elapsed since she took the initial dose.

   Subsequent parts of this question ask students to illustrate particular quantities on their graph, so it's important that students draw their graphs large.




   b. Does the amount of Ibuprofen in Vicki's body (in milligrams) vary at a constant rate with respect to the number of hours since she took the initial dose of 400 mg? Explain.

   Students should say more than, "$f(t)$ is not a linear function of $t$." Push students to explain why $f(t)$ does not vary at a constant rate with respect to $t$ by attending to the multiplicative relationship between $\Delta f(t)$ and $\Delta t$ for various values of $\Delta t$. In other words, students should use the function definition to show that $\Delta f(t)$ and $\Delta t$ are not proportionally related. For example, suppose $t$ changes from $t = 3$ to $t = 4$. Then the corresponding change in $f(t)$ is given by $\Delta f(t) = 400(0.71)^4 - 400(0.71)^3 \approx -41.52$. Suppose $t$ changes from $t = 8$ to $t = 10$. Then the corresponding change in $f(t)$ is given by $\Delta f(t) = 400(0.71)^{10} - 400(0.71)^8 \approx -12.81$. Since $\Delta f(t)$ is not the same number of times as large as $\Delta t$ for different changes in $t$, $\Delta f(t)$ is not proportionally related to $\Delta t$. Therefore, the number of milligrams of Ibuprofen in Vicki's body does not vary at a constant rate with respect to the number of hours elapsed since she took the initial dose of 400 mg. Students could also observe that for successive equal changes in the number of hours elapsed since Vicki took the initial dose of Ibuprofen, corresponding changes in the amount of Ibuprofen in her body increase. (This may not seem to be the case, but remember that the changes in the amount of Ibuprofen in Vicki's body are negative numbers increasing toward zero.) Therefore, $\Delta f(t)$ is not proportionally related to $\Delta t$.

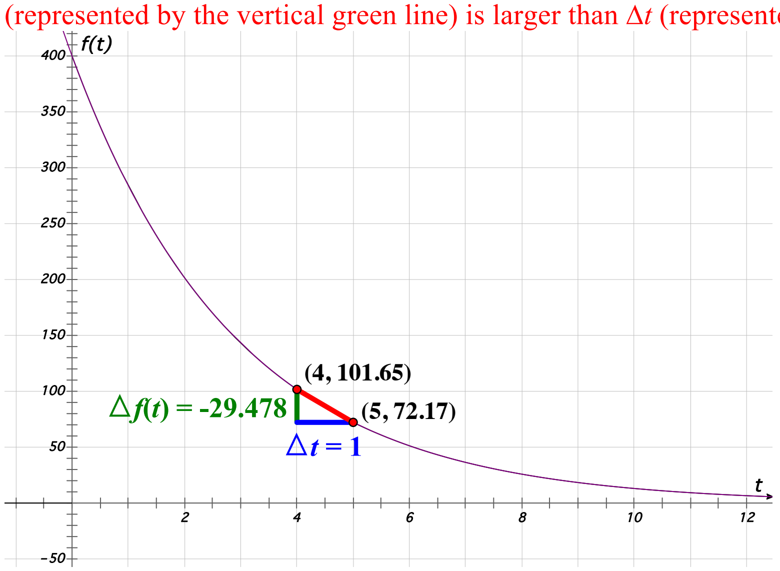
c. Approximate the rate of change of the amount of Ibuprofen in Vicki's body (in milligrams) with respect to the number of hours elapsed since she took the initial dose when $t = 4$. Explain the strategy you used. Also, illustrate what your approximation represents on the graph you drew in Part (a) and explain how what you drew corresponds to the approximation value you computed.

Do not accept qualitative estimates as a response to this question. Encourage students to leverage the meanings developed in previous investigations. In particular, support them in recognizing the need to apply their understanding of average rate of change to approximate the rate of change of the amount of Ibuprofen in Vicki's body (in milligrams) with respect to the number of hours elapsed since she took the initial dose when $t = 4$.

The average rate of change of the amount of Ibuprofen in Vicki's body (in milligrams) with respect to the number of hours elapsed since she took the initial dose over the interval from $t = 4$ to $t = 5$ is an approximation of the rate of change of the amount of Ibuprofen in Vicki's body (in milligrams) with respect to the number of hours elapsed since she took the initial dose when $t = 4$. This average rate of change is given by

$$\frac{400(0.71)^5 - 400(0.71)^4}{5-4} = \frac{-29.478}{1} = -29.478 \text{ mg/hr.}$$

To approximate the rate of change of the amount of Ibuprofen in Vicki's body (in milligrams) with respect to the number of hours elapsed since she took the initial dose when $t = 4$, I computed the average rate of change of the amount of Ibuprofen in Vicki's body (in milligrams) with respect to the number of hours elapsed since she took the initial dose over the interval from $t = 4$ to $t = 5$. This average rate of change is the constant rate of change required for the number of milligrams of Ibuprofen in Vicki's body to decrease by the same amount (29.478 mg) in the same one-hour interval of time. This average rate of change is illustrated as the constant rate of change of the linear function that passes through the points $(4, f(4))$ and $(5, f(5))$ on the graph. This average rate of change is also the slope of the red line on the graph, or the number of times $\Delta f(t)$ (represented by the vertical green line) is larger than $\Delta t$ (represented by the horizontal blue line).




d. Is your approximation from Part (c) an overestimate or underestimate? Justify your response by using language about the physical context, not the shape of the graph.

For successive equal changes in the number of hours elapsed since Vicki took the initial dose of Ibuprofen, corresponding changes in the amount of Ibuprofen in her body increase. The approximation from Part (c) is therefore an overestimate since it represents the average rate of change of the amount of Ibuprofen in Vicki's body (in milligrams) with respect to the number of hours elapsed since she took the initial dose over the interval from $t = 4$ to $t = 5$—a one-hour interval of time immediately beyond the moment of time for which we're approximating the rate of change of the amount of Ibuprofen in Vicki's body (in milligrams) with respect to the number of hours elapsed since she took the initial dose.

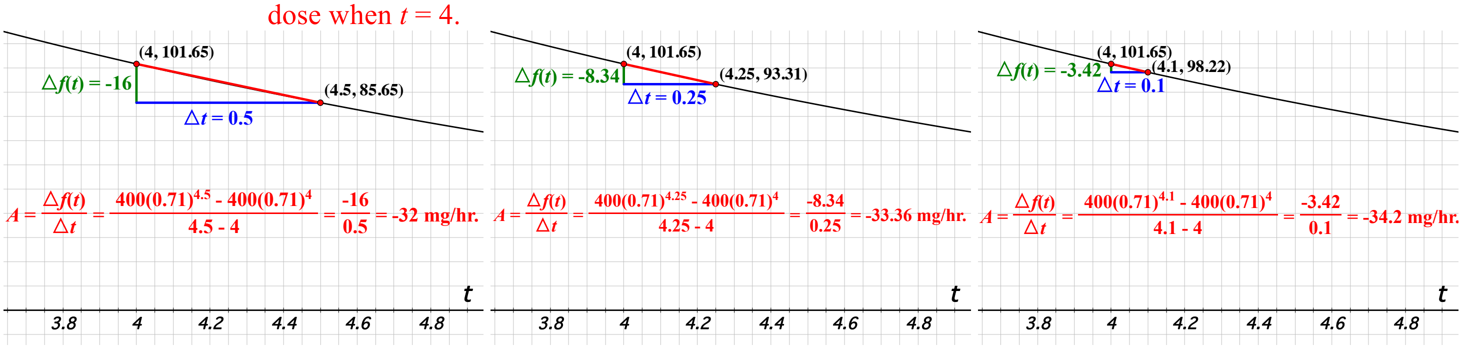
e. Define a variable to represent the value of the quantity you approximated in Part (c). Use this variable to represent the error of your approximation. (Note that the error of an approximation is the positive amount that the approximation differs from the value being approximated.)
Let $R$ represent the rate of change of the amount of Ibuprofen in Vicki's body (in milligrams) with respect to the number of hours elapsed since she took the initial dose. The error of the approximation from Part (c) is represented by either of the following expressions:

$$\left|R - \frac{400(0.71)^5 - 400(0.71)^4}{5-4}\right| \quad \text{or} \quad \frac{400(0.71)^5 - 400(0.71)^4}{5-4} - R.$$

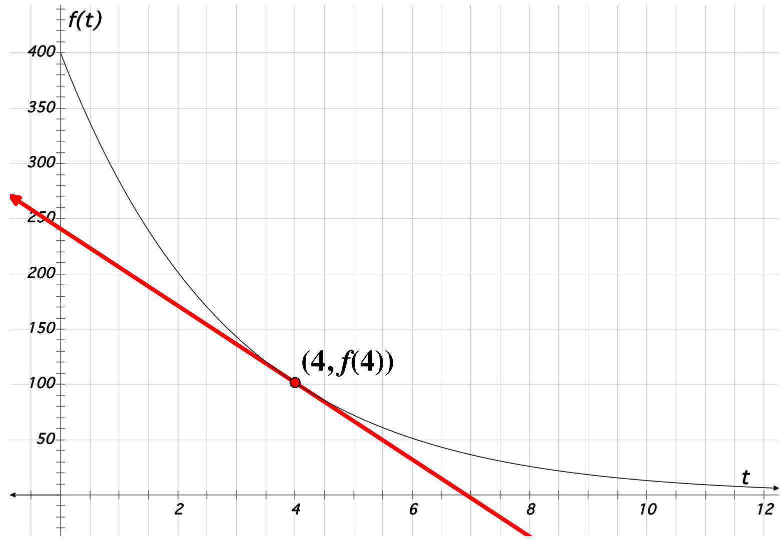
The first expression would accurately represent the error of the approximation even if it were an underestimate, and is for this reason more general than the second. The second expression is correct in this context only because the approximation from Part (c) is an overestimate of the rate of change of the amount of Ibuprofen in Vicki's body (in milligrams) with respect to the number of hours elapsed since she took the initial dose.

f. i. Explain how you might decrease the error of your approximation.
To decrease the error of the approximation, one could decrease the interval of time over which the average rate of change of the amount of Ibuprofen in Vicki's body (in milligrams) with respect to the number of hours elapsed since she took the initial dose is computed. For example, computing the average rate of change of the amount of Ibuprofen in Vicki's body (in milligrams) with respect to the number of hours elapsed since she took the initial dose over the interval from $t = 4$ to $t = 4.5$ would produce a better approximation than the one computed in Part (c).

ii. Compute three approximations that are more accurate than the one you calculated in Part (c) and represent these approximations on three "zoomed in" graphs of the function $f(t) = 400(0.71)^t$.
Let $A$ represent the approximation of the rate of change of the amount of Ibuprofen in Vicki's body (in milligrams) with respect to the number of hours elapsed since she took the initial dose when $t = 4$.




iii. Represent on the graph you drew in Part (a) the value you're approximating and explain how what you drew represents the rate of change of the amount of Ibuprofen in Vicki's body (in milligrams) with respect to the number of hours elapsed since she took the initial dose.

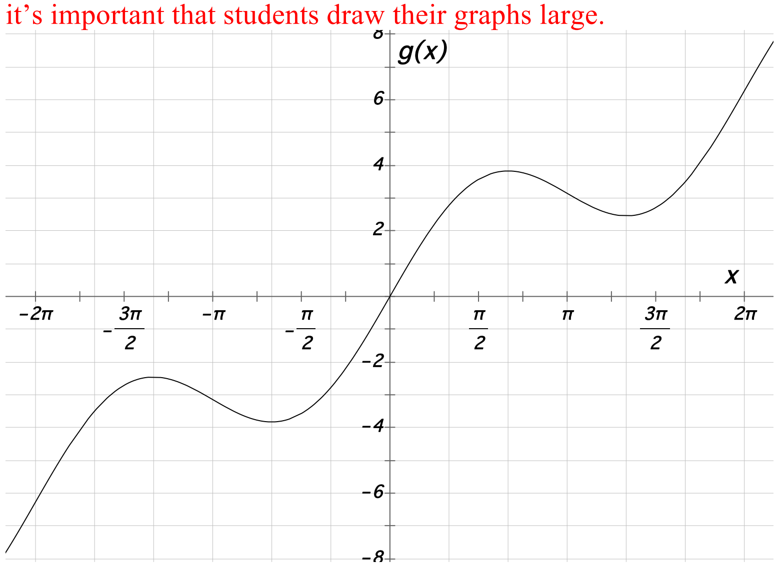



The constant rate of change, or slope, of the linear function tangent to the curve at $t = 4$ (i.e., that passes through the curve only at the point $(4, f(4))$ represents the rate of change of the amount of Ibuprofen in Vicki's body (in milligrams) with respect to the number of hours elapsed since she took the initial dose when $t = 4$. The four approximations we've computed are average rates of change, which geometrically represent the constant rate of change, or slope, of linear functions that pass through two points on the curve, one of them being $(4, f(4))$. We've seen that as we decrease the interval over which we compute these average rates of change, the error of our approximation decreases, and the constant rate of change our approximations represent more closely resemble the constant rate of change of the linear function that is tangent to the curve at $t = 4$. As we make the interval over which we compute the average rate of change of the amount of Ibuprofen in Vicki's body (in milligrams) with respect to the number of hours elapsed since she took the initial dose infinitely small, our approximations become indistinguishable from the constant rate of change of the line tangent to the curve at $t = 4$. A geometric interpretation of the rate of change of the amount of Ibuprofen in Vicki's body (in milligrams) with respect to the number of hours elapsed since she took the initial dose when $t = 4$ is therefore the constant rate of change, or slope, of this tangent line.

iv. Reflect on your responses to Part (ii) and (iii) to determine if there is any point at which it doesn't make sense to decrease the interval over which you're computing the average rate of change of the amount of Ibuprofen in Vicki's body (in milligrams) with respect to the number of hours elapsed since she took the initial dose to get more accurate approximations. In other words, is there any point at which it doesn't make sense to decrease $\Delta t$ to decrease the error of your approximations? Explain. (Feel free to use graphs or equations to communicate your reasoning.)

For very small changes in time around $t = 4$, corresponding changes in the number of milligrams of Ibuprofen in Vicki's body are essentially proportional. Therefore, over a really small interval of time around $t = 4$, the amount of Ibuprofen in Vicki's body (in milligrams) varies at essentially a constant rate with respect to the number of hours elapsed since she took the initial dose of 400 mg. This constant rate of change is indistinguishable from the constant rate of change of the linear function tangent to the curve at $t = 4$, which represents the rate of change of the amount of Ibuprofen in Vicki's body (in milligrams) with respect to the number of hours elapsed since she took the initial dose when $t = 4$. So, if we compute the average rate of change of the amount of Ibuprofen in Vicki's body (in milligrams) with respect to the number of hours elapsed since she took the initial dose over an interval so small that the changes in the input and output quantities are essentially proportional (i.e., the output quantity varies at essentially a constant rate with respect to the input quantity), we gain very little by approximating the rate of change of the amount of Ibuprofen in Vicki's body (in milligrams) with respect to the number of hours elapsed since she took the initial dose when $t = 4$ with an average rate of change over a smaller interval.

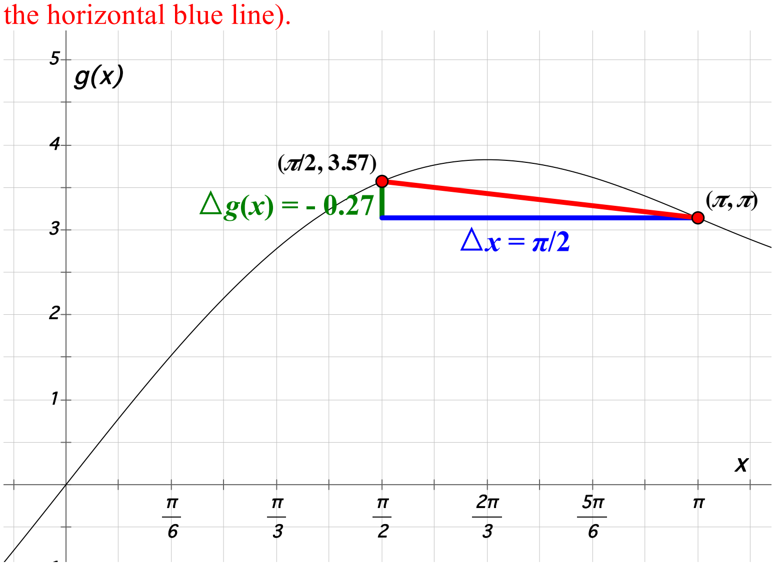
3. Consider the function $g$ defined by $g(x) = 2\sin(x) + x$.
   a. Draw a graph of the function $g$ on the axes provided.
   Subsequent parts of this question ask students to illustrate particular quantities on their graph, so it's important that students draw their graphs large.




   b. Does $g(x)$ vary at a constant rate with respect to $x$? Explain.
   Students should say more than, "$g(x)$ is not a linear function of $x$." Push students to explain why $g(x)$ does not vary at a constant rate with respect to $x$ by attending to the multiplicative relationship between $\Delta g(x)$ and $\Delta x$ for various values of $\Delta x$. In other words, students should use the function definition to show that $\Delta g(x)$ and $\Delta x$ are not proportionally related. For example, suppose $x$ changes from $x = 0$ to $x = \pi/4$. Then the corresponding change in $g(x)$ is given by $\Delta g(x) = 2\sin(\pi/4) + \pi/4 - (2\sin(0) + 0) = 2.2$. Suppose $x$ changes from $x = \pi/4$ to $x = \pi/2$. Then the corresponding change in $g(x)$ is given by $\Delta g(x) = 2\sin(\pi/2) + \pi/2 - (2\sin(\pi/4) + \pi/4) = 1.37$. Since $\Delta g(x)$ is not the same number of times as large as $\Delta x$ for different changes in $x$, $\Delta g(x)$ is not proportionally related to $\Delta x$. Therefore, $g(x)$ does not vary at a constant rate with respect to $x$.
   c. Approximate the rate of change of $g(x)$ with respect to $x$ when $x = \pi/2$ and explain the strategy you used. Also, illustrate what your approximation represents on the graph you drew in Part (a) and explain how what you drew corresponds to the approximation value you computed.
   Do not accept qualitative estimates as a response to this question. Encourage students to leverage the meanings developed in previous investigations. In particular, support them in recognizing the need to apply their understanding of average rate of change to approximate $g$'s rate of change when $x = \pi/2$.
   The average rate of change of $g$ over the interval from $x = \pi/2$ to $x = \pi$ is an approximation of $g$'s rate of change when $x = \pi/2$. This average rate of change is given by

$$\frac{(2\sin(\pi)+\pi)-(2\sin(\pi/2)+\pi/2)}{\pi-\pi/2}=\frac{-0.43}{\pi/2}=-0.27.$$

To approximate the rate of change of $g(x)$ with respect to $x$ at $x = \pi/2$, I computed the car's average rate of change over the interval $x = \pi/2$ to $x = \pi$. This average rate of change is the constant rate of change that achieves the same change in output as the function $g$ over the interval $[\pi/2, \pi]$. This average rate of change represents the slope of the red line on the graph, or the number of times $\Delta g(x)$ (represented by the vertical green line) is larger than $\Delta x$ (represented by the horizontal blue line).



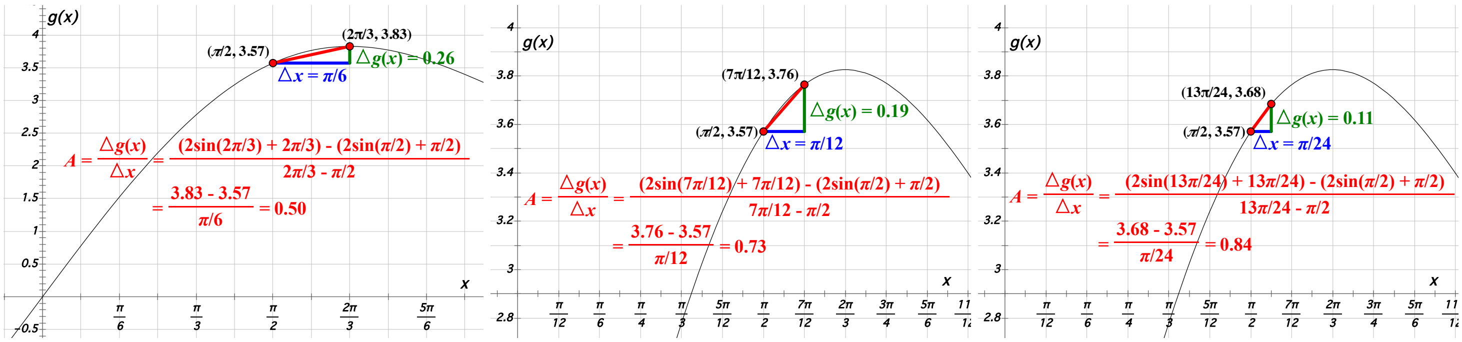

d. Is your approximation from Part (c) an overestimate or underestimate? Justify your response by using language about changes in the input and output variables, not the shape of the graph.
In a local "neighborhood" around $x = \pi/2$ we notice that for successive equal changes in $x$, the corresponding changes in $g(x)$ decrease. The approximation from Part (c) is therefore an underestimate since it represents the average rate of change of the function $g$ from $x = \pi/2$ to $x = \pi$—an interval of the input variable immediately beyond the value of $x$ at which we're approximating $g$'s rate of change.

e. Define a variable to represent the value of the quantity you approximated in Part (c). Use this variable to represent the error of your approximation. (Note that the error of an approximation is the positive amount that the approximation differs from the value being approximated.)
Let $A$ represent the rate of change of $g(x)$ with respect to $x$ when $x = \pi/2$. The error of the approximation from Part (c) is represented by:

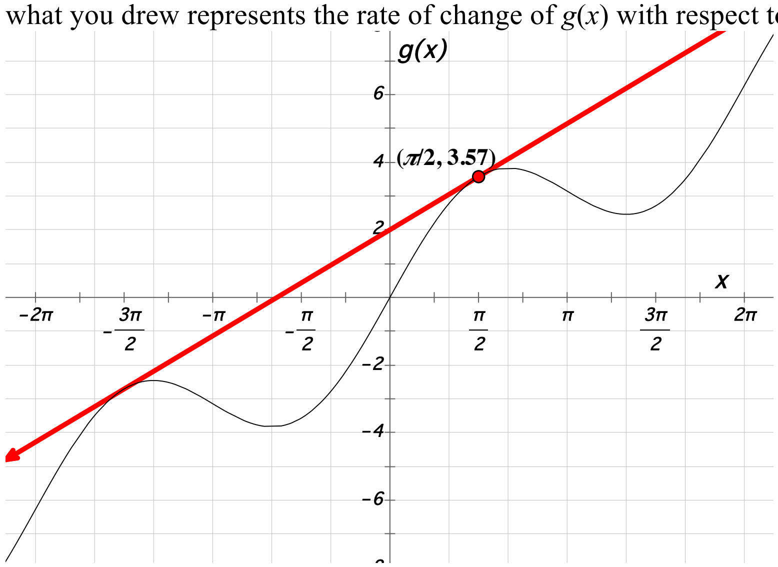
$$\left| A - \frac{(2\sin(\pi)+\pi)-(2\sin(\frac{\pi}{2})+\frac{\pi}{2})}{\pi-\frac{\pi}{2}} \right|.$$

f. i. Explain how you might decrease the error of your approximation.
To decrease the error of the approximation, one could decrease the interval of the input variable over which the average rate of change of $g(x)$ with respect to $x$ is computed. For example, computing $g$'s average rate of change over the interval from $x = \pi/2$ to $x = 3\pi/2$ would produce a better approximation than the one computed in Part (c).

ii. Compute three approximations that are more accurate than the one you calculated in Part (c) and represent these approximations on three "zoomed in" graphs of the function $g(x) = 2\sin(x) + x$.
Let $A$ represent the approximation of the rate of change of $g(x)$ with respect to $x$ when $x = \pi/2$.




iii. Represent on the graph you drew in Part (a) the value you're approximating and explain how what you drew represents the rate of change of $g(x)$ with respect to $x$ when $x = \pi/2$.




The constant rate of change, or slope, of the linear function tangent to the curve at $x = \pi/2$ (i.e., that passes through the curve only at the point $(\pi/2, g(\pi/2))$ represents the rate of change of $g(x)$ with respect to $x$ when $x = \pi/2$. The four approximations we've computed are average rates of change, which geometrically represent the constant rate of change, or slope, of linear functions that pass through two points on the curve, one of them being $(\pi/2, g(\pi/2))$. We've seen that as we decrease the interval over which we compute these average rates of change, the error of our approximation decreases, and the constant rate of change our approximations represent more closely resemble the constant rate of change of the linear function that is tangent to the curve at $x = \pi/2$. As we make the interval over which we compute $g$'s average rate of change infinitely small, our average rate of change approximations become indistinguishable from the constant rate of change of the line tangent to the curve at $x = \pi/2$. A geometric interpretation of the rate of change of $g(x)$ with respect to $x$ when $x = \pi/2$ is therefore the constant rate of change, or slope, of this tangent line.

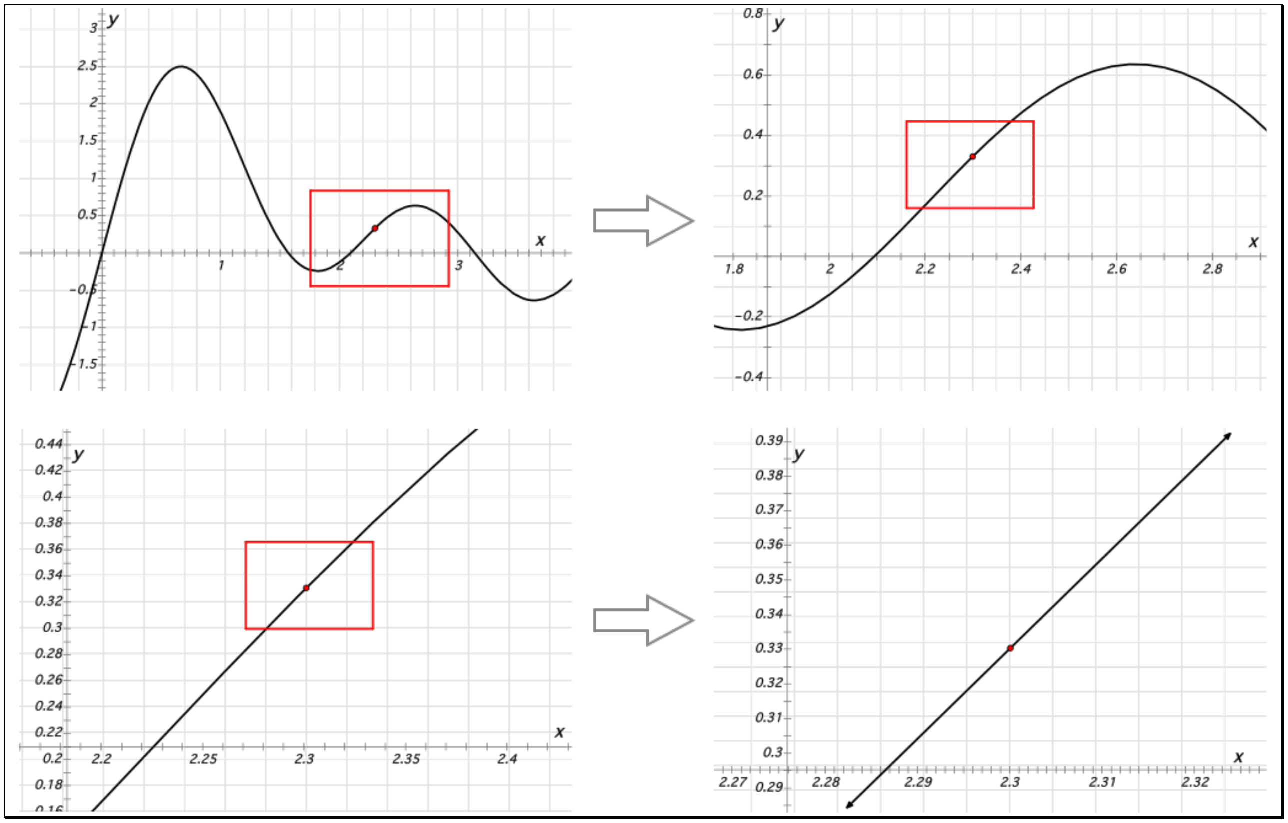
iv. Reflect on your responses to Part (ii) and (iii) to determine if there is any point at which it doesn't make sense to decrease the interval over which you're computing the average rate of change of the function $g$ to approximate the rate of change of $g(x)$ with respect to $x$ when $x = \pi/2$. In other words, is there any point at which it doesn't make sense to decrease $\Delta x$ to get more accurate approximations? Explain. (Feel free to use graphs or equations to communicate your reasoning.)

For very small changes in $x$ around $x = \pi/2$, corresponding changes in $g(x)$ are essentially proportional. Therefore, over a really small interval of the input variable around $x = \pi/2$, $g(x)$ varies at essentially a constant rate with respect to $x$. This constant rate of change is indistinguishable from the constant rate of change of the linear function tangent to the curve at $x = \pi/2$, which represents the rate of change of $g(x)$ with respect to $x$ when $x = \pi/2$. So, if we compute $g$'s average rate of change over an interval so small that the changes in the input and output quantities are essentially proportional (i.e., the output quantity varies at essentially a constant rate with respect to the input quantity), we gain very little by approximating the rate of change of $g(x)$ with respect to $x$ when $x = \pi/2$ with an average rate of change over a smaller interval.

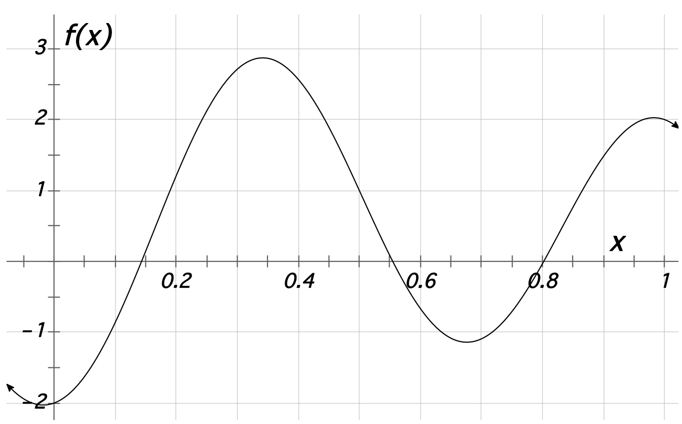
In Calculus, we often refer to the ***instantaneous rate of change*** of one quantity with respect to another. This is slightly misleading since, as we have seen, a rate of change is a multiplicative comparison of *changes* in quantities' values. The rate of change of Quantity A with respect to Quantity B is the number of times a change in the measure of Quantity A is larger than the corresponding change in the measure of Quantity B. Rates of change therefore do not occur at an instant—*they require changes in quantities' measures to exist!*

Consider the context in Problem 1. The only way we could determine the car's speed 8 seconds after it started moving was to approximate it by computing an average rate of change over a very small interval of time around $t = 8$. Without a small change in time, there is no corresponding change in distance, and thus no rate of change. Since rates of change always occur over an interval of the input variable (even really small intervals), you should interpret "instantaneous rate of change" as "average rate of change over an interval so small that the changes in the quantities' measures are essentially proportional." The input and output quantities vary essentially at a constant rate over these very small intervals, making the graphs look linear. This concept is referred to as ***local linearity***, or ***local constant rate of change***. It is important to note that local linearity does not always occur. We will examine a function that is not locally linear in Problem 6.

Suppose that $y = f(x)$ is function. We say that the function $f$ is ***locally linear*** near an input value $x = a$ if $y$ varies at essentially a constant rate with respect to $x$ near $x = a$ (i.e., $\Delta y$ is essentially proportional to $\Delta x$ near $x = a$). If $f$ is locally linear near $x = a$, the graph of $f$ looks increasingly like a straight line the closer we zoom in on the point $(a, f(a))$. This straight line is called the *tangent line* to the graph of $f$ at $x = a$. The specified point $(a, f(a))$ is called the *point of tangency*. The following is an example of a function that is locally linear near the input value $x = 2.3$. As we zoom in closer and closer on the point (2.3, 0.3305), the graph of the function starts to look like a straight line, which suggests that over very small intervals of the input variable around $x = 2.3$, $y$ varies at essentially a constant rate with respect to $x$.



4. The following is a graph of $f(x) = \sin(\pi x) - 2\cos(3\pi x)$ on the input interval [0, 1].
   Students tend to have difficulty understanding that the graph of the tangent line should approximate certain aspects of the graph of the function near the point of tangency. In other words, the "local linearity" concept and its implications are not well-established in the students' minds. This task supports students in making the connection between the slope of the tangent line and approximations to that slope via average rates of change.




   a. Enter the formula for the function $f$ into your graphing calculator. (Make sure your calculator is in radian mode.) Use the zoom feature on the calculator to zoom in on the point (0.4, $f$(0.4)) until the graph of $f$ looks like a straight line.

It will take three zooms to obtain a reasonably straight line. Students need to make sure the cursor remains on the graph with each zoom.

b. Use your graph from Part (a) to estimate the local constant rate of change of $f(x)$ with respect to $x$ near $x = 0.4$. Explained how you determined this rate of change.
Most graphing utilities do not include grid lines, and axes probably will not appear on the final zoom. It is therefore not obvious how to estimate the local constant rate of change of $f(x)$ with respect to $x$ near $x = 0.4$. In this case, students will have no choice but to estimate additional points on the curve and use these points to estimate this rate of change. Students can use the explicit formula or the zoom-in to estimate a second point. The following are questions for the groups to consider:
   - What decimal place accuracy should be used for the points on the zoom-in? [Probably no more than three decimal place accuracy based on issues with the zoom.]
   - Is it important to keep the change in input value small?

   It is now appropriate to further emphasize the true meaning of the number the students are computing (average rate of change for the curve on a specified input interval.

   Using the point (0.40, 2.569) and the point (0.41, 2.461), the average rate of change for the function on the input interval $0.40 \leq x \leq 0.41$ is approximately $-10.8$.

c. Construct a formula for the tangent line to the graph of $f$ at the point $(0.4, f(0.4))$.
Encourage students to use point-slope form since it makes direct use of the estimated slope and the given point. The point-slope formula for the tangent line is $y = L(x) \approx -10.8(x - 0.40) + 2.569$.

d. Starting with your final zoom-in, graph the function $f$ along with the tangent line to the graph of $f$ you constructed in Part (c). What do you notice about the two graphs? Now, start zooming out. What do you notice as you return to the original viewing window $0 \leq x \leq 1$? Carefully draw what you see on the graph provided above.
Students' answers as to what they notice will vary. Listen to responses from the groups and bring interesting responses to the attention of the whole class. If no one is doing so, suggest that groups examine the relationship between the graph of the function and the graph of the tangent line. It is important for students to understand that the graph of the tangent line "looks like" the graph of the function near the point of tangency. More importantly, students should recognize that the constant rate of change of the tangent line is indistinguishable from the local constant rate of change of the function $f$ near $x = 0.4$.

e. Do you think it is possible to keep zooming in forever on the graph of $f$ at the point $(0.4, f(0.4))$? What problems might you encounter while attempting to do so? Are these problems with your graphing device or are they problems with the function itself? Explain.
Listen carefully to the discussions that arise from this question. Students often imagine that "zooming in" forever is mathematically impossible, in large part because they struggle to understand functions as a concept distinct from the image appearing on their calculator screen or paper. Students often say that zooming in "too much" will make the graph blurry, or will cause the graph to pixelate. Ask groups to consider what it means to say that the specified function is independent of the physical limitations of the graphing calculator or the drawn graph. Take some ideas from the class, but don't push this too far. Depending on the timbre of the discussions, you can suggest that students think about how they could describe "zooming in" mathematically, without relying on a graphing calculator.

f. Without using the zoom feature on your graphing calculator, estimate the local constant rate of change of $f(x)$ with respect to $x$ near $x = 0.2$ and $x = 0.6$. How did you obtain your estimates?
If students understand the relationship between the graph of the tangent line and the function near the point of tangency (see Part (d) above), they should have no trouble estimating the local

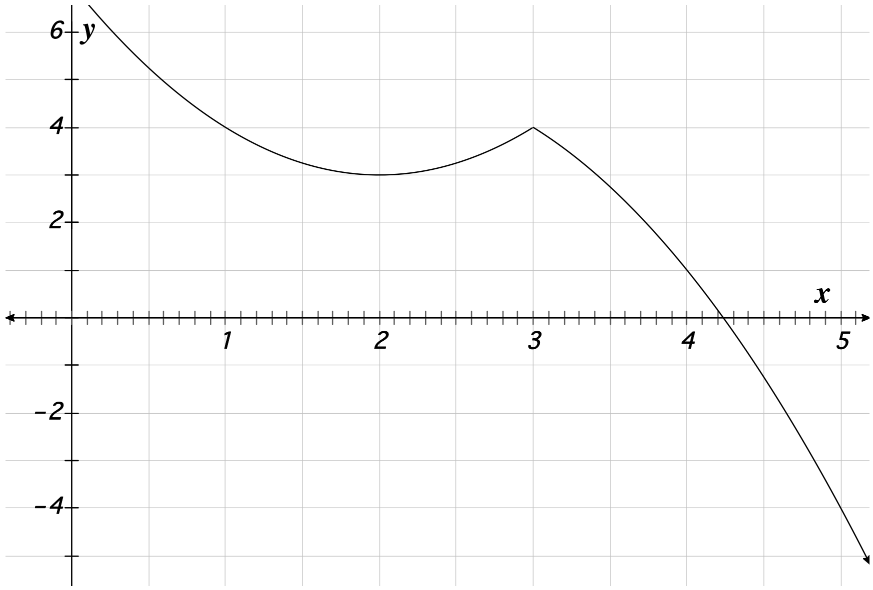
constant rate of change of $f(x)$ with respect to $x$ near $x = 0.2$ and $x = 0.6$ by sketching and approximating the slope of tangent lines.

5. Suppose the function $f$ is locally linear near the input value $x = a$. Using ideas previously discussed in this course, write an expression that represents the instantaneous rate of change of $f(x)$ with respect to $x$ at $x = a$. Explain what your expression represents without using the word "instantaneous."
   Use this problem, and the discussion of notation that follows, to introduce the definition of derivative at $x = a$.
   The instantaneous rate of change of $f(x)$ with respect to $x$ is given by
   $$\lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}.$$
   This expression represents the limiting value of the average rate of change of $f(x)$ with respect to $x$ from $x = a$ to $x = a + \Delta x$ as $\Delta x$, the length of the interval over which the average rate of change is computed, approaches zero.

The expression you wrote in response to Problem 5 defines what is called the ***derivative of f with respect to x at x = a*** and represents the limiting value of the average rate of change of $f(x)$ with respect to $x$ from $x = a$ to $x = a + \Delta x$ as $\Delta x$, the length of the interval over which the average rate of change is computed, approaches zero. We use two notations to represent this quantity: $\left.\frac{df}{dx}\right|_{x=a}$ and $f'(a)$. The first notation is read, "dee $f$ dee $x$ evaluated at $x = a$" and clearly resembles $\Delta f/\Delta x$. Recall that $\Delta x$ represents a change in $x$ and $\Delta f$ represents the corresponding change in $f(x)$. The symbols "$dx$" and "$df$" still refer to changes in the measures of the input and output quantities, but we use $d$ instead of $\Delta$ to denote that these changes are so small that they are essentially proportionally related. The second notation, $f'(a)$, is read "$f$ prime of $a$." We have two different notations to represent the same quantity because calculus was simultaneously developed by two individuals, Isaac Newton and Gottfried Leibnitz, who represented their ideas differently.

6. Consider the function $f(x) = 1 + |x - 1|$.
   a. Enter this function into your graphing calculator and use your calculator to sketch its graph in the viewing window $-1 \le x \le 2$, $0 \le y \le 3$. Do you think that the function $f$ is locally linear at the input value $x = 0$? Explain.
      Students should recognize that the $f$ is locally linear at $x = 0$, but might have a hard time explaining why. Encourage them to phrase their justification in terms of the definition above. Strive for a statement like, "When we zoom in really close to the point (0, 1), the changes in the input and output quantities appear proportionally related (i.e., the rate at which $f(x)$ varies with respect to $x$ appears constant)."
   b. Is the function $f$ locally linear near the input value $x = 1$? Explain.
      Again, strive for a response that appeals to the definition of local linearity. For example, strive for a statement like, "When we zoom in really close to the point (1, 1), the changes in the input and output quantities are clearly not proportionally related (i.e., the rate at which $f(x)$ varies with respect to $x$ is clearly not constant)."

7. Consider the graph of the function $y = f(x)$ shown below.




a. Estimate the values of $f'(1)$ and $f'(2)$ and explain how you determined your estimates.
Encourage students to estimate the values of $f'(1)$ and $f'(2)$ by respectively estimating the slopes of the lines tangent to the graph of the function $f$ at $x = 1$ and $x = 2$. In particular, encourage students to sketch tangent lines to the graph of $f$ at $x = 1$ and $x = 2$ and to compute their constant rates of change. Hopefully, students will recognize that the tangent line to the graph at the point (2, 2) is horizontal and therefore has a constant rate of change of zero; however, students often do not trust their intuition and will compute the slope anyway.

b. Is it possible to estimate the value of $f'(3)$? If so, estimate this value. If not, explain why.
The "pointiness" of the graph of $f$ suggests that the function does not have a local constant rate of change near the point (3, 3). Therefore, $f'(3)$ does not exist.

**Homework**

1. Use your graphing calculator to sketch a graph of the function

$$f(x) = \begin{cases} 1 + x^2, & x \le 1 \\ 3x - 1, & x > 1 \end{cases}$$

Is $f$ locally linear at the input value $x = 1$? Explain your reasoning. (You may need to look up methods for graphing piecewise-defined functions on your calculator.)
The function $f$ is not locally linear at the input value $x = 1$. In other words, $f(x)$ does not vary at essentially a constant rate with respect to $x$ over *any* small interval including $x = 1$. However, this only becomes apparent after several zooms centered on the point (1, 2).

2. Is the function $f(x) = \sqrt[3]{x}$ locally linear at the input value $x = 0$? Explain your reasoning.
Student responses will vary, since the definition of local linearity does not directly preclude the possibility that the graph will come to look more and more like a *vertical* line as we zoom in. However, since vertical lines have no definable slope and do not represent functions of the input variable, the understanding among mathematicians is that such behavior does not constitute local linearity. Encourage concerned students to modify the definition to clarify this ambiguity.

3. Let $f(x) = \sqrt[3]{x}$.

a. Use the zoom feature on your graphing calculator to estimate the value of $f'(2)$.
Students' responses will vary somewhat, but should be close to $f'(2) \approx 0.21$.

b. Define a linear function $L$ that is tangent to the graph of $f$ at the point $\left(1, \sqrt[3]{2}\right)$.
$L(x) \approx 0.21(x-2)+\sqrt[3]{2}$.

c. Using your current zoom window, graph the tangent line along with the function $f$. What do you notice?
Students should notice that the graph of their tangent line looks almost exactly like the graph of the function $f$ in the viewing window. Encourage students to phrase this observation in terms of inputs and outputs.

d. Change the viewing window on your calculator to $-1 \le x \le 8$ and $-1 \le y \le 2$. What do you notice?
Students should notice that the graph of their tangent line looks like the graph of the function $f$ near the point of tangency, but looks less like the graph of the function far from the point of tangency. Encourage students to phrase this observation in terms of inputs and outputs.

4. Let $g(x) = \ln(x)$.

a. Use the zoom feature on your graphing calculator to estimate the value of $g'(1)$.
Students' responses will vary somewhat, but should be close to $g'(1) = 1$.

b. Define a linear function $L$ that is tangent to the graph of $g$ at the point (1, 0).
$L(x) \approx 1(x-1)+0 = x-1$.

c. Using your current zoom window, graph the tangent line along with the function $g$. What do you notice?
Students should notice that the graph of their tangent line looks almost exactly like the graph of the function $f$ in the viewing window. Encourage students to phrase this observation in terms of inputs and outputs.

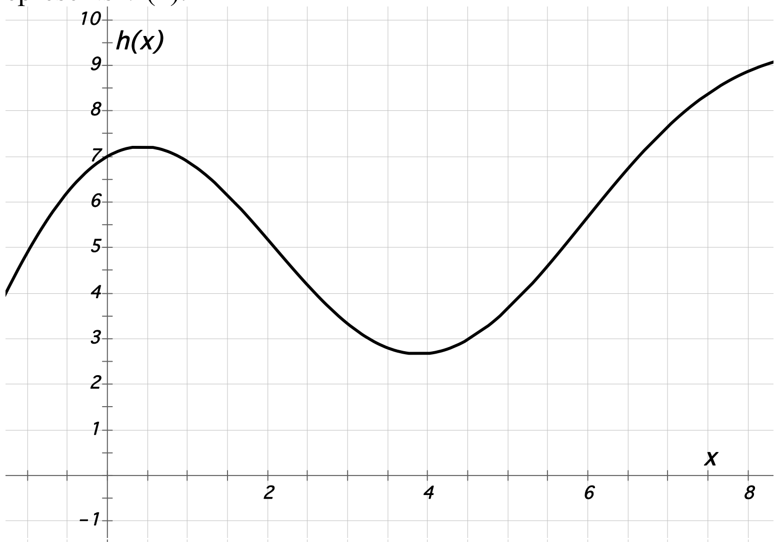
d. Change the viewing window on your calculator to $0 \le x \le 4$ and $-5 \le y \le 2$. What do you notice?
Students should notice that the graph of their tangent line looks like the graph of the function $f$ near the point of tangency, but looks less like the graph of the function far from the point of tangency. Encourage students to phrase this observation in terms of inputs and outputs.

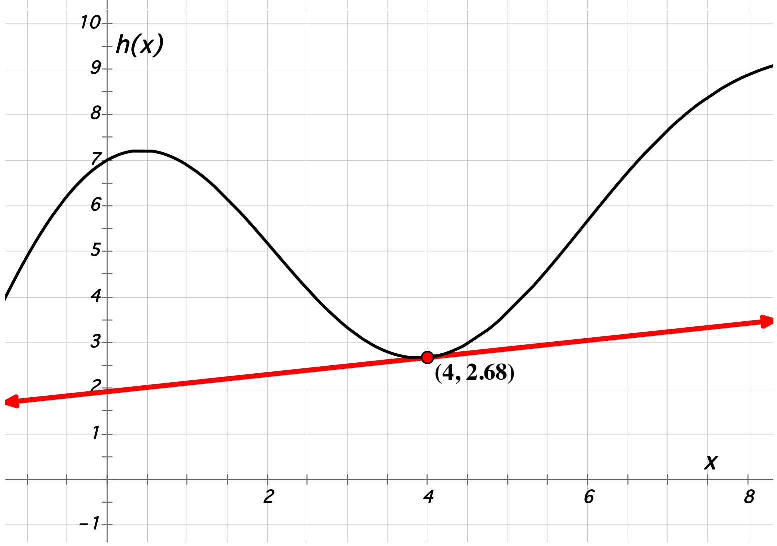
5. Imagine a bottle filling with water. Let $x$ represent the height of the water in the bottle (in centimeters) and let $g(x)$ represent the volume of water in the bottle (in milliliters). Explain the meaning of $\left.\frac{dg}{dx}\right|_{x=3.7}$. Do not use the word "instantaneous" in your explanation.

The value $\left.\frac{dg}{dx}\right|_{x=3.7}$ represents the limiting value of the average rates of change of volume of water in the bottle (in milliliters) over the interval $[3.7, 3.7 + \Delta x]$ as $\Delta x$, the interval of the water's height in the bottle (in centimeters) over which the average rate of change is computed, approaches zero. More generally, the value $\left.\frac{dg}{dx}\right|_{x=3.7}$ represents the constant rate of change of $g(x)$ with respect to $x$ over a very small interval around $x = 3.7$.

6. Let $w$ represent the age of a basset hound puppy (in weeks) and let $f(w)$ represent the puppy's weight (in pounds). Explain the meaning of $f'(15)$. Do not use the word "instantaneous" in your explanation.
The value $f'(15)$ represents the limiting value of the average rates of change of the puppy's weight (in pounds) with respect to the puppy's age (in weeks) over the interval $[15, 15 + \Delta w]$ as $\Delta w$, the change in the puppy's age from 15 weeks, approaches zero. More generally, the value $f'(15)$ represents the constant rate of change of the puppy's weight (in pounds) with respect to the puppy's age (in weeks) over a very small interval around $w = 15$.

7. The following is a graph of the function $h$. Represent on this graph the value $h'(4)$ and explain how what you drew represents $h'(4)$.




The slope of the line tangent to the graph of the function $h$ when $x = 4$ represents the value $h'(4)$. Because $h$ is locally linear near $x = 4$, the local constant rate of change of $h(x)$ with respect to $x$ near $x = 4$ is essentially the same as the constant rate of change represented by the slope of the line tangent to the graph of $h$ that passes through the point (4, 2.68).

8. Let $f(x) = \sin(x) + \sin(2x)$. Compute $f'(7.3)$. Explain what your solution represents.
$f'(7.3)$ represents the local constant rate of change of the function $f(x)$ with respect to $x$ near $x = 7.3$. In other words, $f'(7.3)$ represents the limiting value of the average rate of change of $f(x)$ with respect to $x$ over the interval $[7.3, 7.3 + \Delta x]$ as $\Delta x$, the interval over which the average rate of change is computed, approaches zero. Using a graphing program, we notice that the graph of the function $f$ appears locally linear over the interval [7.3, 7.30001]. Therefore, the constant rate of change of $f(x)$ with respect to $x$ near $x = 7.3$ is given by

$$\frac{f(7.30001) - f(7.3)}{7.30001 - 7.3} = \frac{\sin(7.30001) + \sin(14.60002) - (\sin(7.3) + \sin(14.6))}{0.00001} = 0.05109.$$

9. Let $g(t) = t^2 - 3t + 1$. Compute $\left.\frac{dg}{dt}\right|_{t=-2.2}$. Explain what your solution represents.

$\left.\frac{dg}{dt}\right|_{t=2.2}$ represents the local constant rate of change of the function $g(t)$ with respect to $t$ near $t = 2.2$. In other words, $\left.\frac{dg}{dt}\right|_{t=2.2}$ represents the limiting value of the average rate of change of $g(t)$ with respect to $t$ over the interval $[2.2, 2.2 + \Delta t]$ as $\Delta t$, the interval over which the average rate of change is computed, approaches zero. Using a graphing program, we notice that the graph of the function $g$ appears locally linear over the interval [2.2, 2.2001]. Therefore, the constant rate of change of $g(t)$ with respect to $t$ near $t = 2.2$ is given by

$$\frac{g(2.2001)-g(2.2)}{2.2001-2.2} = \frac{2.2001^2 - 3(2.2001)+1-(2.2^2-3(2.2)+1)}{0.0001} = 1.4001.$$

10. Let $h(r) = 1.7^r - \cos(r)$. Compute $h'(5.9)$. Explain what your solution represents.

$h'(5.9)$ represents the local constant rate of change of the function $h(r)$ with respect to $r$ near $r = 5.9$. In other words, $h'(5.9)$ represents the limiting value of the average rate of change of $h(r)$ with respect to $r$ over the interval $[5.9, 5.9 + \Delta r]$ as $\Delta r$, the interval over which the average rate of change is computed, approaches zero. Using a graphing program, we notice that the graph of the function $h$ appears locally linear over the interval [5.9, 5.90005]. Therefore, the constant rate of change of $h(r)$ with respect to $r$ near $r = 5.9$ is given by

$$\frac{h(5.90005)-h(5.9)}{5.90005-5.9} = \frac{1.7^{5.90005} - \cos(5.90005) - (1.7^{5.9} - \cos(5.9))}{0.00005} = 12.1481.$$

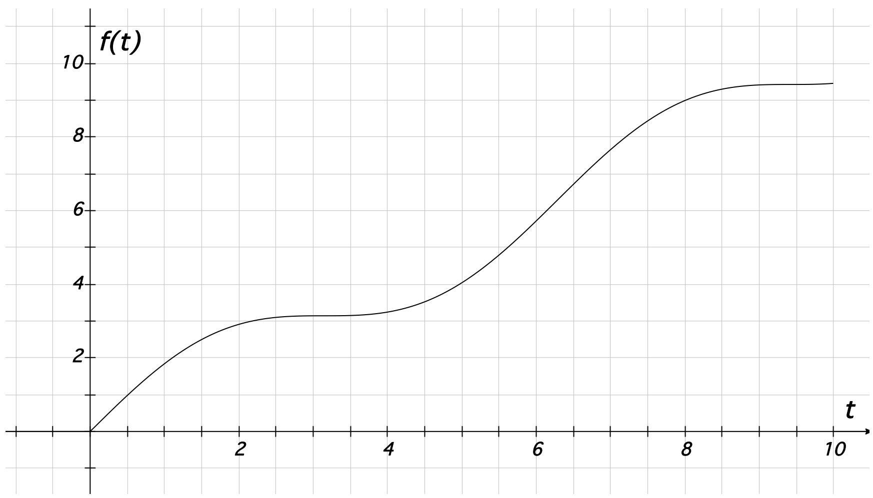
11. Let $j(p) = \ln(p)$. Compute $\left.\frac{dj}{dp}\right|_{p=3.8}$. Explain what your solution represents.

$\left.\frac{dj}{dp}\right|_{p=3.8}$ represents the local constant rate of change of the function $j(p)$ with respect to $p$ near $p = 3.8$. In other words, $\left.\frac{dj}{dp}\right|_{p=3.8}$ represents the limiting value of the average rate of change of $j(p)$ with respect to $p$ over the interval $[3.8, 3.8 + \Delta p]$ as $\Delta p$, the interval over which the average rate of change is computed, approaches zero. Using a graphing program, we notice that the graph of the function $j$ appears locally linear over the interval [3.8, 3.8001]. Therefore, the constant rate of change of $j(p)$ with respect to $p$ near $p = 3.8$ is given by

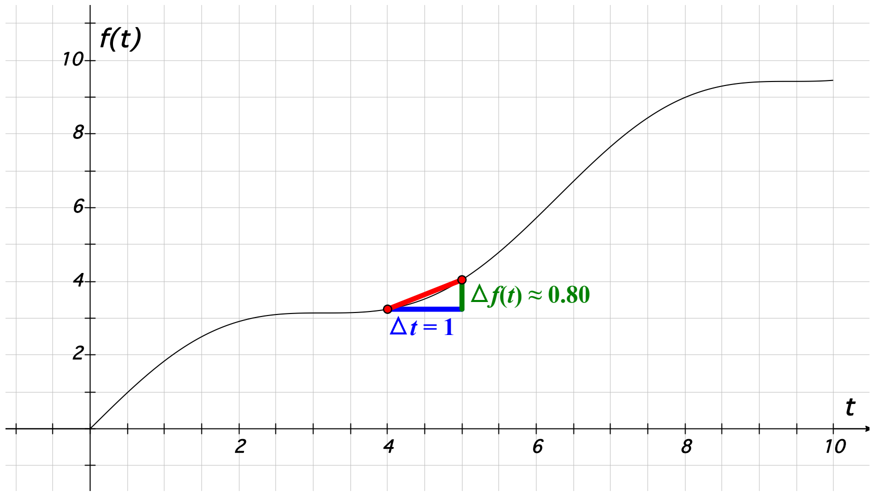
$$\frac{j(3.8001)-j(3.8)}{3.8001-3.8} = \frac{\ln(3.8001)-\ln(3.8)}{0.0001} = 0.2632.$$

The concept of a function is fundamental to the study of mathematics. Functions relate two covarying quantities and thus enable one to predict outcomes (the measure of a dependent quantity) based on given information (the measure of an independent quantity). That is, for respective independent and dependent quantities $x$ and $y$, one can ask, "What value of $y$ corresponds to a particular value of $x$?" The usefulness of functions, however, extends beyond comparing the two covarying quantities related in a function definition. For respective independent and dependent quantities $x$ and $y$, one can also ask, "What is the rate at which $y$ is changing for specific values of $x$?" For this question to be answered for a continuum of $x$ values, it is necessary to generate a new function for which the dependent quantity is the rate at which $y$ is changing for specific $x$ values. This function is called a derivative function of $y$ with respect to $x$. The purpose of this investigation is to examine how the derivative function is generated and understand the information it conveys. We leverage the meanings developed in previous investigations by continuously asking students to interpret symbolic and graphical representations of derivative functions in terms of rates of change.

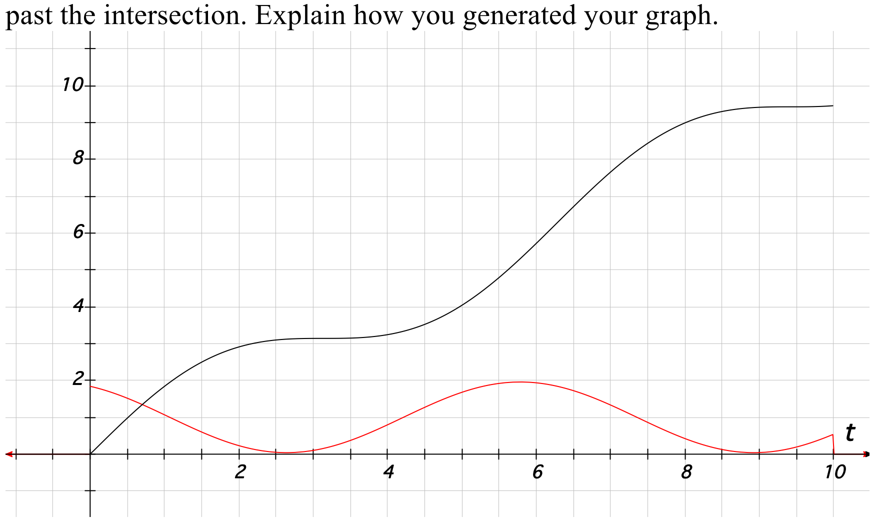
1. The graph below represents the relationship between a car's distance in kilometers from an intersection (represented by $f(t)$) and the number of minutes elapsed since the car passed the intersection (represented by the variable $t$).




   a. Approximate the average rate of change of $f(t)$ with respect to $t$ over the interval [4, 5] and illustrate the value of your approximation on the graph above. Explain what your approximation represents in the context of this situation.
   As $t$ changes from 4 to 5, the corresponding change in $f(t)$ is approximately 0.80. Therefore, the average rate of change of $f(t)$ with respect to $t$ over the interval [4, 5] is approximately ${}^{0.8}/_{1} = 0.8$. This average rate of change represents the average speed of the car from 4 to 5 minutes since the car passed the intersection. This average speed represents the constant speed at which another car would have to travel to cover the same distance as the original car in the one-minute interval of time between $t = 4$ and $t = 5$. The slope of the red secant line on the graph below represents the average speed of the car from 4 to 5 minutes after it passed the intersection.



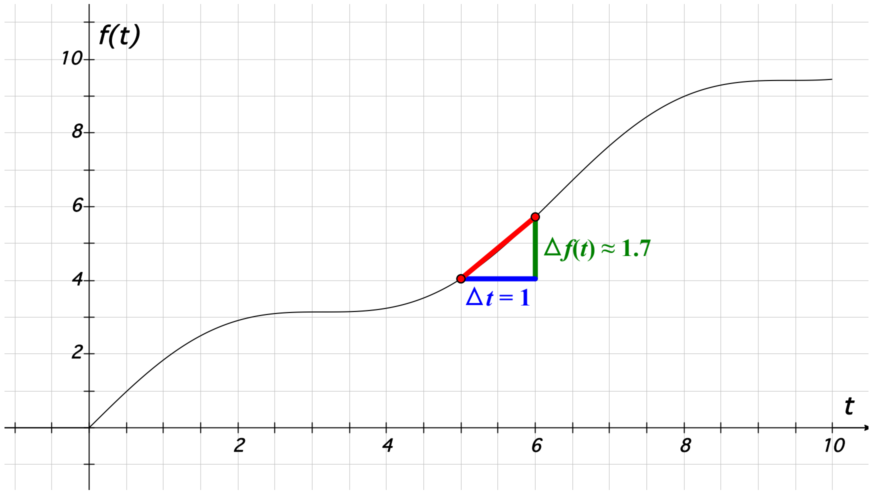

b. Sketch a graph (as accurately as possible) that represents the relationship between the average speed of the car over *any* one-minute interval and the number of minutes elapsed since the car past the intersection. Explain how you generated your graph.




If students are struggling to generate the graph, ask them to represent the average rate of change of $f(t)$ with respect to $t$ over the interval [0, 1] on the graph of $f$ and to plot this average rate of change on the axes provided. Then ask students to imagine varying $t$ from 0 to 10 and prompt them to describe how the average rate of change of $f(t)$ with respect to $t$ changes as $t$ varies. Then ask them to capture this variation in their graph.

c. Pick a point on the graph you sketched in Part (b). Explain the meaning of the point's coordinates and illustrate this meaning on the graph that represents the relationship between $f(t)$ and $t$.

The point (5, 1.7) appears to be on the graph of the function sketched in Part (b). The coordinates of this point convey that the car had an average speed of 1.7 kilometers per minute over the interval of time from 5 to 6 minutes after the car passed the intersection. The slope of the red secant line on the graph below represents this average speed.




d. Define a function $r_f$, which stands for "rate of change of $f$," in terms of $f(t)$ that determines the average speed of the car over the interval $[t, t + \Delta t]$, where $\Delta t$ is some positive constant. Explain the meaning of $r_f(6)$.

$$r_f(t) = \frac{f(t+\Delta t) - f(t)}{\Delta t}.$$

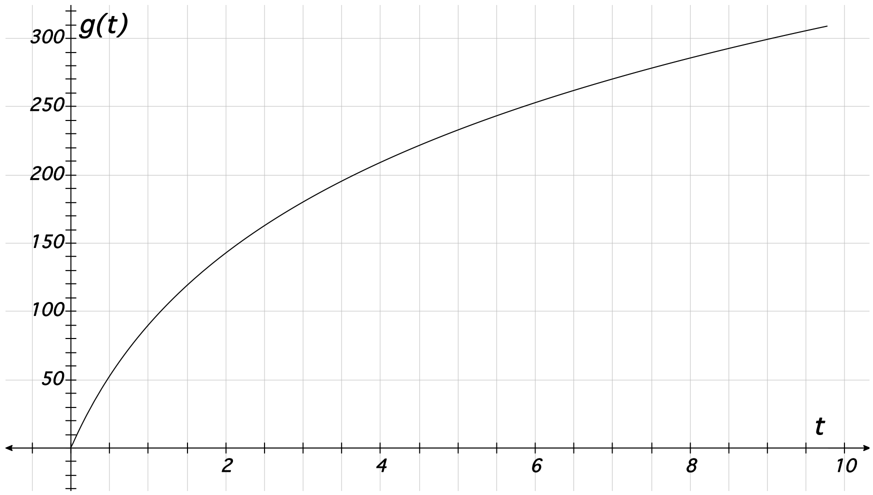
$r_f(6)$ represents the average speed of the car from 6 to 6 + $\Delta t$ minutes after the car passed the intersection.

e. Define a function $f'$ in terms of $f(t)$ that determines the "instantaneous" speed of the car $t$ minutes after it passed the intersection. Explain the meaning of $f'(6)$.

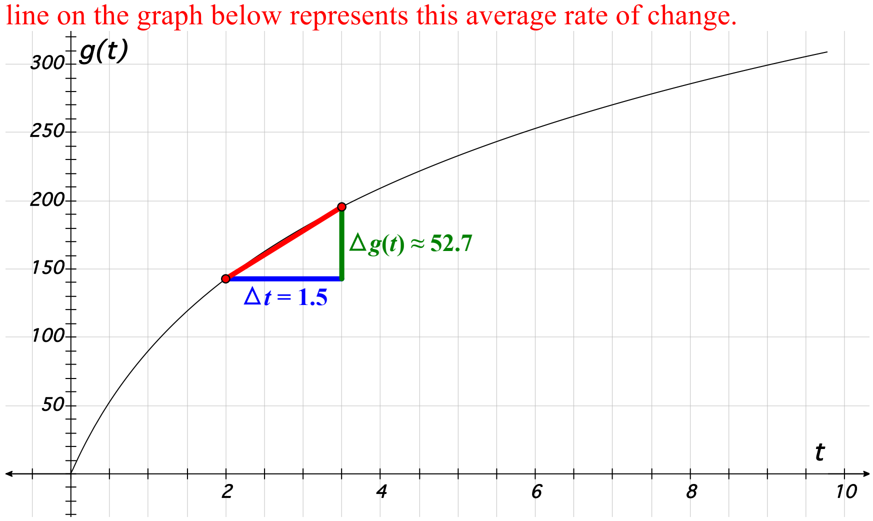
$$f'(t) = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}.$$

$f'(6)$ represents the limiting value of the average speed of the car from 6 to 6 + $\Delta t$ minutes after the car passed the intersection as $\Delta t$, the interval of time over which the average rate of change is computed, approaches zero. In other words, $f'(6)$ represents the essential constant speed of the car over very small intervals of time that begin or end at $t = 6$.

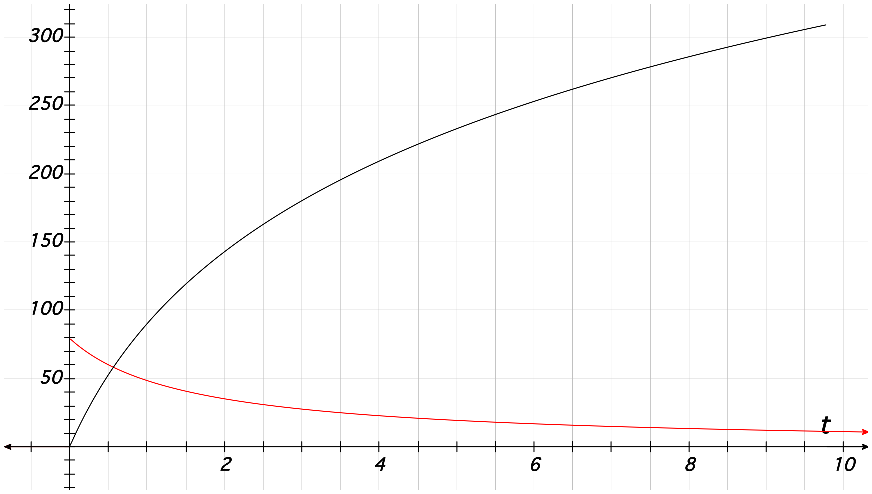
2. Phil Mickelson, a professional golfer, hit a drive at the 2016 Open Championship that flew 309 yards in the air. The graph below represents the relationship between the ball's horizontal distance in yards from where it was struck (represented by $g(t)$) and the number of seconds elapsed since Phil hit the ball (represented by the variable $t$).




a. Approximate the average rate of change of $g(t)$ with respect to $t$ over the interval [2, 3.5] and illustrate the value of your approximation on the graph of $g$. Explain what your approximation represents in the context of this situation.
As $t$ changes from 2 to 3.5, the corresponding change in $g(t)$ is approximately 52.7. Therefore, the average rate of change of $g(t)$ with respect to $t$ over the interval [2, 3.5] is approximately ${}^{52.7}/_{1.5} \approx$ 35.13. This value represents the average rate of change of the ball's horizontal distance (in yards) from Phil Mickelson over the interval of time from $t = 2$ to $t = 3.5$. The slope of the red secant line on the graph below represents this average rate of change.

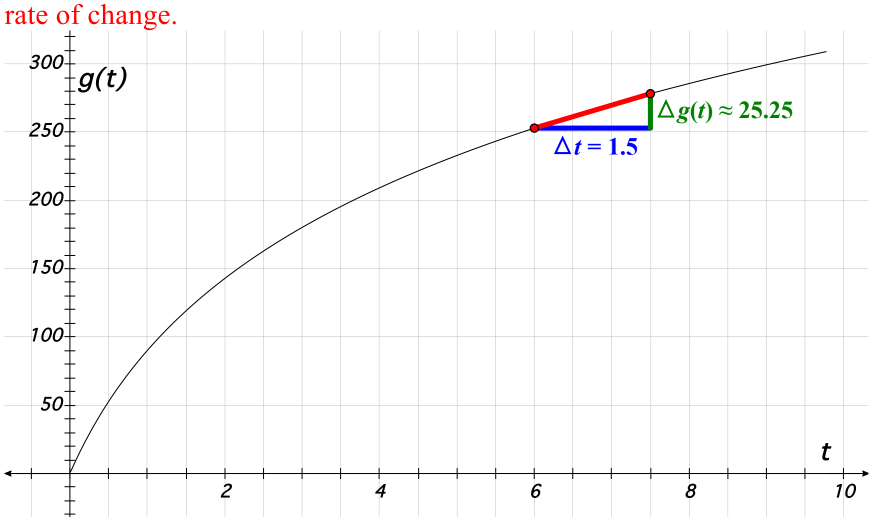



b. Sketch a graph (as accurately as possible) that represents the relationship between the following quantities and explain how you generated your graph.
- *Dependent quantity*: The average rate of change of the ball's horizontal distance (in yards) from Phil Mickelson over *any* 1.5-second interval.
- *Independent quantity*: The number of seconds elapsed since Phil hit the ball.




If students are struggling to generate the graph, ask them to represent the average rate of change of $g(t)$ with respect to $t$ over the interval [0, 1.5] on the graph of $g$ and to plot this average rate of change on the axes provided. Then ask students to imagine varying $t$ from 0 to 9.5 and prompt them to describe how the average rate of change of $g(t)$ with respect to $t$ changes as $t$ varies. Then ask them to capture this variation in their graph.

c. Pick a point on the graph you sketched in Part (b). Explain the meaning of the point's coordinates and illustrate this meaning on the graph that represents the relationship between $d$ and $t$.
The point (6, 16.8) appears to be on the graph of the function sketched in Part (b). The coordinates of this point convey that the average rate of change of the ball's horizontal distance (in yards) from Phil Mickelson over the interval of time from $t = 6$ to $t = 7.5$ is approximately 16.8 yards per second. The slope of the red secant line on the graph below represents this average rate of change.




d. Define a function $r_g$, which stands for "rate of change of $g$," in terms of $g(t)$ that determines the average rate of change of $g(t)$ with respect to $t$ over the interval $[t, t + \Delta t]$, where $\Delta t$ is some positive constant. Explain the meaning of $r_g(8.2)$.

$$r_g(t) = \frac{g(t+\Delta t) - g(t)}{\Delta t}.$$

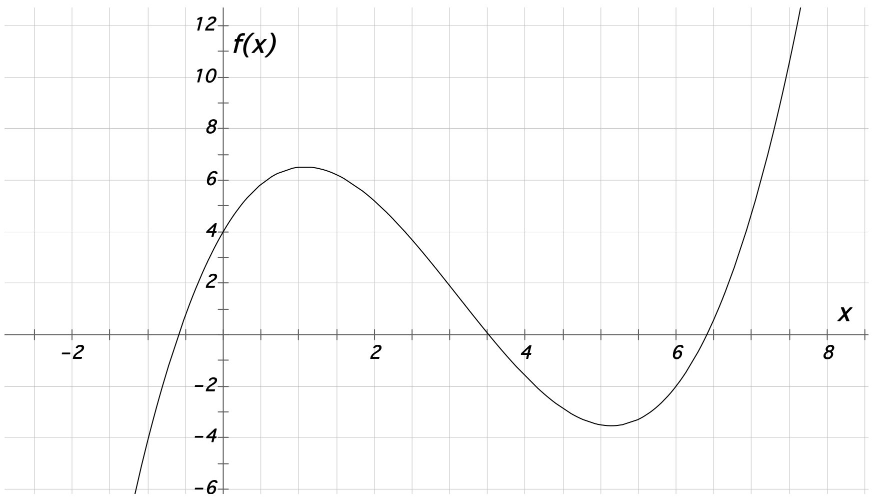
$r_g(8.2)$ represents the average rate of change of the ball's horizontal distance (in yards) from Phil Mickelson from 8.2 to $8.2 + \Delta t$ seconds after he struck the ball.

e. Define a function $g'$ in terms of $g(t)$ that determines the "instantaneous" rate of change of the ball's horizontal distance (in yards) from Phil Mickelson with respect to the number of seconds elapsed since he hit the ball. Explain the meaning of $g'(8.2)$.

$g'(t) = \lim_{\Delta t \to 0} \dfrac{g(t+\Delta t) - g(t)}{\Delta t}$.

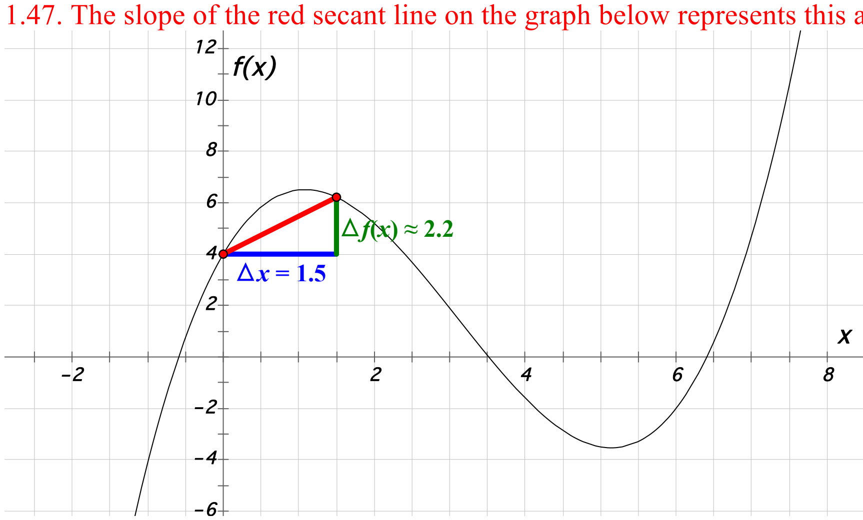
$g'(8.2)$ represents the limiting value of the average rate of change of the ball's horizontal distance (in yards) from Phil Mickelson from 8.2 to 8.2 + $\Delta t$ seconds after he struck the ball as $\Delta t$, the interval of time over which the average rate of change is computed, approaches zero. In other words, $g'(8.2)$ represents the essential constant rate of change of the ball's horizontal distance (in yards) from Phil Mickelson over a very small interval of time that begins or ends at $t = 8.2$.

3. Consider the following graph of the function $f$.

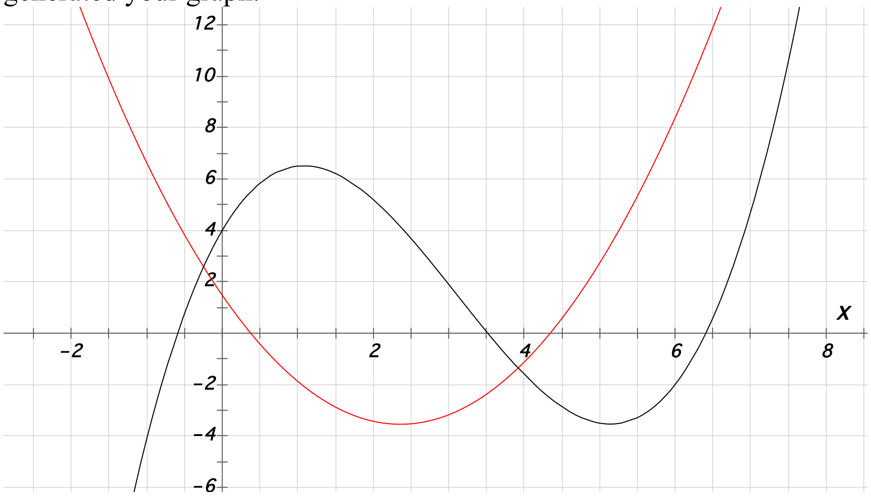



a. Approximate the average rate of change of $f(x)$ with respect to $x$ over the interval [0. 1.5] and illustrate the value of your approximation on the graph above. Explain what your approximation represents in the context of this situation.

As $x$ changes from 0 to 1.5, the corresponding change in $f(x)$ is approximately 2.2. Therefore, the average rate of change of $f(x)$ with respect to $x$ over the interval [0, 1.5] is approximately ${}^{2.2}/_{1.5} \approx$ 1.47. The slope of the red secant line on the graph below represents this average rate of change.



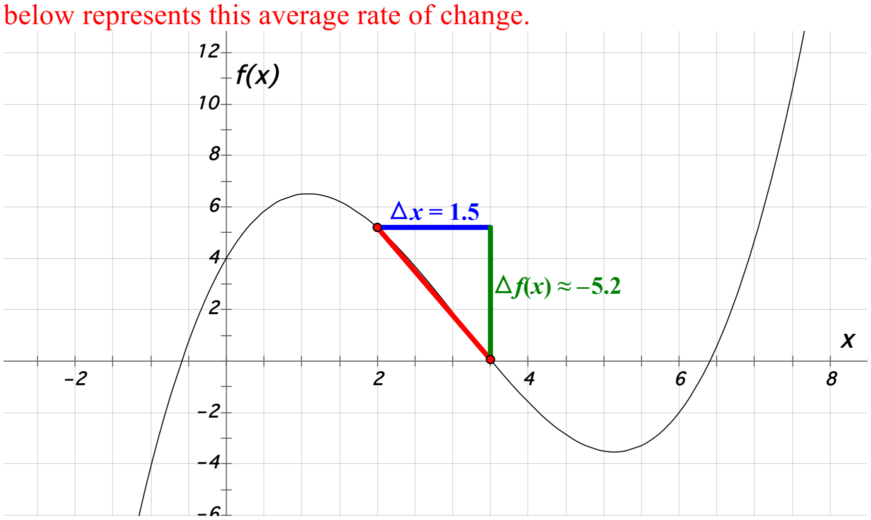
b. Sketch a graph (as accurately as possible) that represents the relationship between the average rate of change of $f(x)$ with respect to $x$ of the car over the interval $[x, x + 1.5]$. Explain how you generated your graph.




If students are struggling to generate the graph, ask them to represent the average rate of change of $f(x)$ with respect to $x$ over the interval $[0, 1.5]$ on the graph of $f$ and to plot this average rate of change on the axes provided. Then ask students to imagine varying $x$ from –2 to 8 and prompt them to describe how the average rate of change of $f(x)$ with respect to $x$ changes as $x$ varies. Then ask them to capture this variation in their graph.

c. Pick a point on the graph you sketched in Part (b). Explain the meaning of the point's coordinates and illustrate this meaning on the graph that represents the relationship between $f(x)$ and $x$.

The point (2, –3.5) appears to be on the graph of the function sketched in Part (b). The coordinates of this point convey that the average rate of change of $f(x)$ with respect to $x$ over the interval from $t = 2$ to $t = 3.5$ is approximately –3.5. The slope of the red secant line on the graph below represents this average rate of change.




d. Define a function $r_f$, which stands for "rate of change of $f$," in terms of $f(x)$ that determines the average rate of change of $f(x)$ with respect to $x$ over the interval $[x, x + \Delta x]$ (for positive $\Delta x$) or $[x + \Delta x, x]$ (for negative $\Delta x$). Explain the meaning of $r_f(4.7)$.

$$r_f(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

$r_f(4.7)$ represents the average rate of change of $f(x)$ with respect to $x$ over the interval from $x = 4.7$ to $x = 4.7 + \Delta x$ if $\Delta x$ is positive, or the interval from $x = 4.7 + \Delta x$ to $x = 4.7$ if $\Delta x$ is negative.

e. Define a function $f'$ in terms of $f(x)$ that determines the "instantaneous" rate of change of $f(x)$ with respect to $x$. Explain the meaning of $f'(4.7)$.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

$f'(4.7)$ represents the limiting value of the average rate of change of $f(x)$ with respect to $x$ over the interval $[4.7, 4.7 + \Delta x]$ as $\Delta x$, the interval over which the average rate of change is computed, approaches zero. In other words, $f'(4.7)$ represents the essential constant rate of change of $f(x)$ with respect to $x$ over a very small interval of time that begins or ends at $x = 4.7$.

The functions you defined in Part (e) of Problems 1-3 is called a ***derivative function***, or simply ***derivative***. The derivative of a function $f$ is the function $f'$ whose outputs represent the limiting value of the average rate of change of $f(x)$ with respect to $x$ over the interval $[x, x + \Delta x]$ (for positive $\Delta x$) or $[x + \Delta x, x]$ (for negative $\Delta x$) as $\Delta x$, the length of the interval over which the average rate of change is computed, approaches zero. In other words, $f'(x)$ represents the local constant rate of change of $f(x)$ with respect to $x$. The derivative of the function $f$ is given by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

provided the limit exists.

**Homework**

1. Let $f$ be a function. Suppose we define $r_f$ and $f'$ as follows:

$$r_f(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

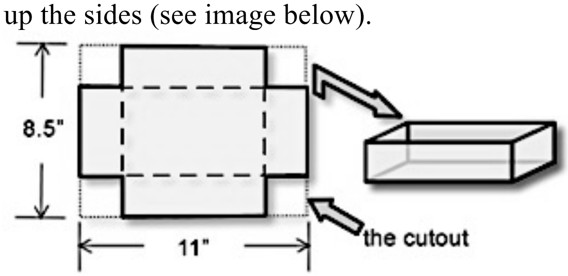
For a value $x = a$ in the domain of $f$, $r_f$, and $f'$, explain what each expression represents.

a. $f(a)$: $f(a)$ represents the output value of the function $f$ at $x = a$.

b. $r_f(a)$: $r_f(a)$ represents the average rate of change of $f(x)$ with respect to $x$ over the interval $[a, a + \Delta x]$.

c. $f'(a)$: $f'(a)$ represents the limiting value of the average rate of change of $f(x)$ with respect to $x$ over the interval $[a, a + \Delta x]$ as $\Delta x$ approaches zero. In other words, $f'(x)$ represents the local constant rate of change of $f(x)$ with respect to $x$ at $x = a$.

2. An open-top box can be created by cutting four equal-sized square corners from an 8.5 by 11-inch sheet of paper and folding up the sides (see image below).



a. Define a function $f$ that determines the volume of the box (in cubic inches) provided the length of the side of the square cutout $x$ (in inches). Express the polynomial in both standard and factored form.
Factored form: $f(x) = x(11 - 2x)(8.5 - 2x)$
Standard form: $f(x) = 4x^3 - 39x^2 + 93.5x$

b. Compute $f'(x)$ using the definition of the derivative function.

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{[4(x+\Delta x)^3 - 39(x+\Delta x)^2 + 93.5(x+\Delta x)] - (4x^3 - 39x^2 + 93.5x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{[4x^3 + 12x^2\Delta x + 12x\Delta x^2 + 4\Delta x^3 - 39x^2 - 78x\Delta x - 39\Delta x^2 + 93.5x + 93.5\Delta x] - (4x^3 - 39x^2 + 93.5x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{12x^2\Delta x + 12x\Delta x^2 + 4\Delta x^3 - 78x\Delta x - 39\Delta x^2 + 93.5\Delta x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta x(12x^2 + 12x\Delta x + 4\Delta x^2 - 78x - 39\Delta x + 93.5)}{\Delta x} \\
&= \lim_{\Delta x \to 0} (12x^2 + 12x\Delta x + 4\Delta x^2 - 78x - 39\Delta x + 93.5) \\
&= 12x^2 - 78x + 93.5
\end{aligned}$$

c. Evaluate $f'(0.6)$ and explain what this value represents in the context of this situation.
$f'(0.6) = 12(0.6)^2 - 78(0.6) + 93.5 = 51.02$.
This means that the limiting value of the average rate of change of the volume of the box (in cubic centimeters) with respect to the length of the side of the cutout over the interval $[0.6, 0.6 + \Delta x]$ if $\Delta x$ is positive, or the interval $[0.6 + \Delta x, 0.6]$ if $\Delta x$ is negative, as $\Delta x$ approaches zero is 51.02 $cm^3/cm$. In other words, the local constant rate of change of $f(x)$ with respect to $x$ over a very small interval that begins or ends at $x = 0.6$ is 51.02 $cm^3/cm$.

3. Suppose a baseball outfielder fields a ball and throws it back towards the infield, releasing it from his hand 6.5 feet above ground level at an angle of 18º above the horizontal at a speed of 103 feet per second. Neglecting air resistance, the baseball's height above the ground $h$ (in feet) after $t$ seconds since it was released can be modeled by the function $g(t) = -16t^2 + 31.829t + 6.5$.

a. Compute $g'(t)$ using the definition of the derivative function.

$$\begin{aligned}
g'(t) &= \lim_{\Delta t \to 0} \frac{[-16(t+\Delta t)^2 - 31.829(t+\Delta t) + 6.5] - (-16t^2 - 31.829t + 6.5)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{(-16t^2 - 32t\Delta t - 16\Delta t^2 - 31.829t - 31.829\Delta t + 6.5) - (-16t^2 - 31.829t + 6.5)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{-32t\Delta t - 16\Delta t^2 - 31.829\Delta t}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{\Delta t(-32t - 16\Delta t - 31.829)}{\Delta t} \\
&= \lim_{\Delta t \to 0} (-32t - 16\Delta t - 31.829) \\
&= -32t - 31.829
\end{aligned}$$

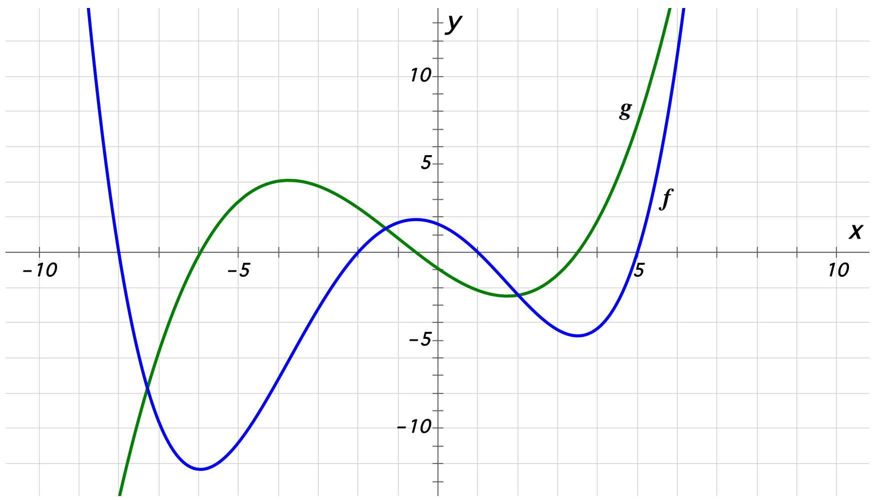
b. Evaluate $g'(1.7)$ and explain what this value represents in the context of this situation.
$g'(1.7) = -32(1.7) - 31.829 = -86.229$.
This means that the limiting value of the average rate of change of the ball's height above the ground (in feet) with respect to the number of seconds elapsed since the ball was released over the interval $[1.7, 1.7 + \Delta t]$ if $\Delta t$ is positive, or the interval $[1.7 + \Delta t, 1.7]$ if $\Delta t$ is negative, as $\Delta t$

approaches zero is –86.229 ft/sec. In other words, the local constant rate of change of $g(t)$ with respect to $t$ over a very small interval that begins or ends at $t = 1.7$ is –86.229 ft/sec.

For exercises 4-9, compute $f'(x)$ using the definition of the derivative function.

4. $f(x) = x^2 - 1$
   $f'(x) = 2x$
5. $f(x) = 7x + 2$
   $f'(x) = 7$
6. $f(x) = 3x^2 + 2x$
   $f'(x) = 6x + 2$
7. $f(x) = 8.4x - 19$
   $f'(x) = 8.4$
8. $f(x) = x^3$
   $f'(x) = 3x^2$
9. $f(x) = 4 - 7x^3$
   $f'(x) = -21x^2$

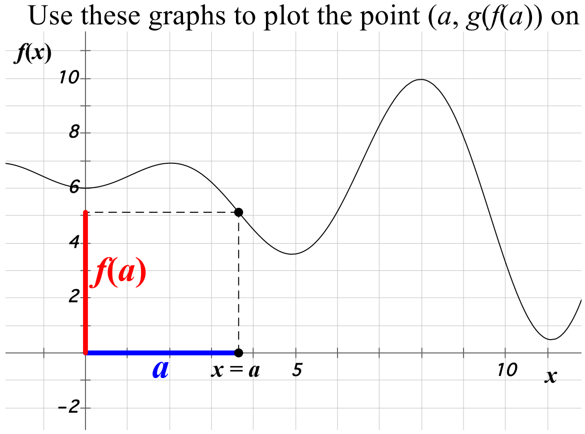
10. The functions $f$ and $g$ are graphed on the same axes below. Determine which function is the derivative of the other. Explain your reasoning.
    For each value of $x$ in the domain of the functions $f$ and $g$ shown in the graph, the output values $g(x)$ represent the local constant rate of change of $f(x)$ with respect to $x$ over very small intervals around $x$. The function $g$ is therefore the derivative of the function $f$.

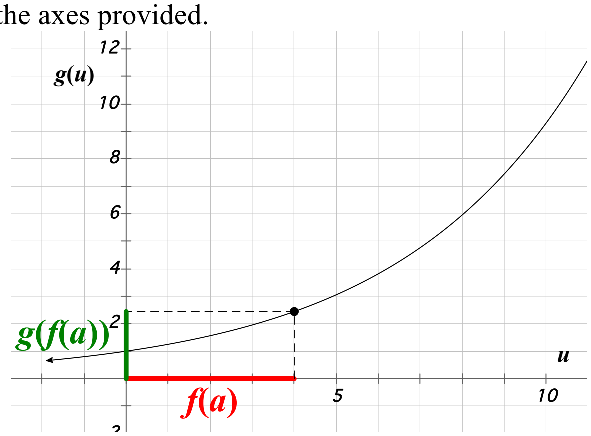


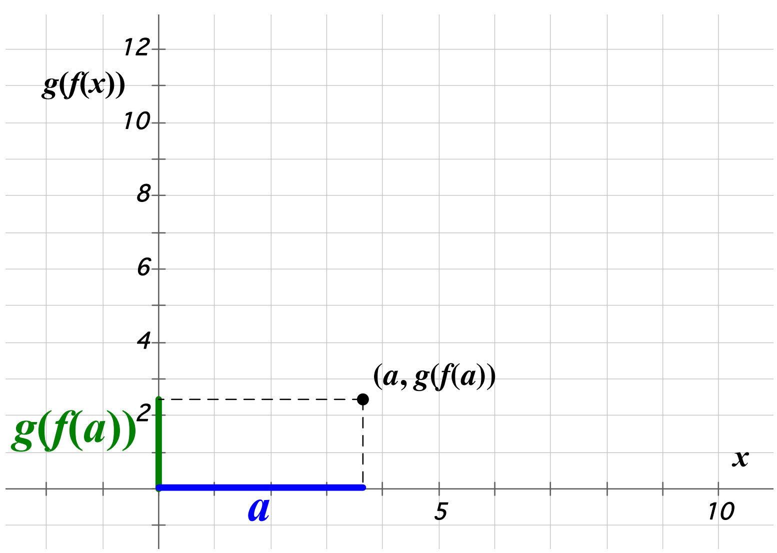
Problem 1 of this investigation provides an opportunity for students to review the graphical interpretation of function composition. The rest of this investigation heavily relies upon the understandings supported in Problem 1, so review this problem carefully with students before proceeding. Problems 2-4 guide students through the process of representing the average rate of change of various composite functions in context. The purpose of these problems is to allow students to abstract the structure of determining the average rate of change of a composite function in a way that provides a conceptual foundation for the chain rule. Problem 5 prompts students to review their work on Problems 2-4 to generate a conjecture for how to represent the average rate of change of the generic composite function $(f \circ g)(x)$ with respect to $x$. Students verify their conjectures in Problem 6. The investigation concludes with a formalization of students' representation of the average rate of change of the generic composite function $(f \circ g)(x)$ with respect to $x$ from Problem 6. This formalization, which results from applying a limit, results in the chain rule. We do not include a homework section since it would not differ substantially from what appears in standard calculus curricula.

The understandings promoted in this investigation were informed by the content in Chapter 6 of *Calculus: Newton Meets Technology* by Patrick Thompson, Mark Ashbrook, Stacy Musgrave, and Fabio Milner (Thompson et al., 2015).

1. The graphs of the functions $f$ and $g$ are provided below.
   a. Use these graphs to plot the point $(a, g(f(a))$ on the axes provided.

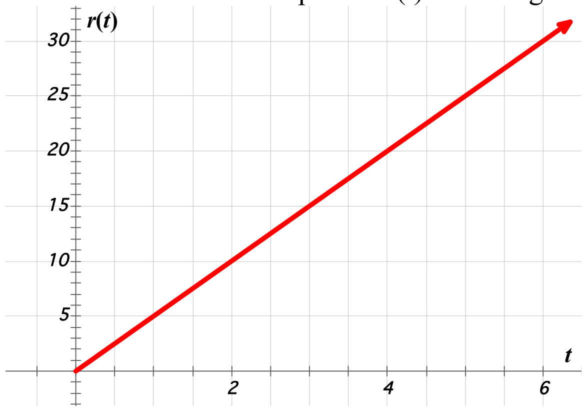


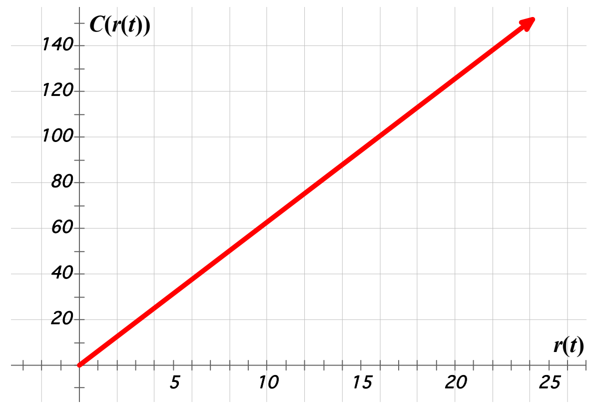



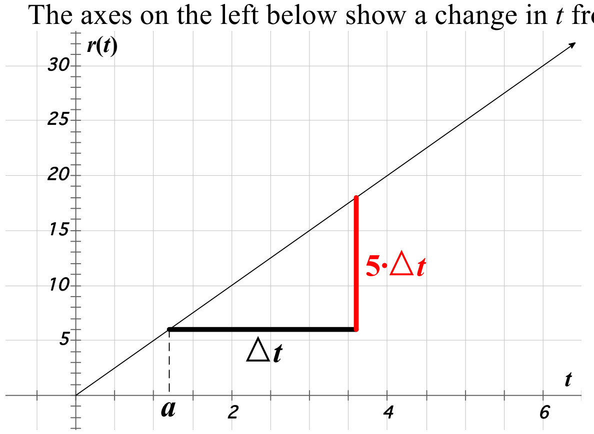

   b. Use the graphs of $f$ and $g$ above to approximate the value of $g(f(9))$. $g(f(9)) \approx 6$.

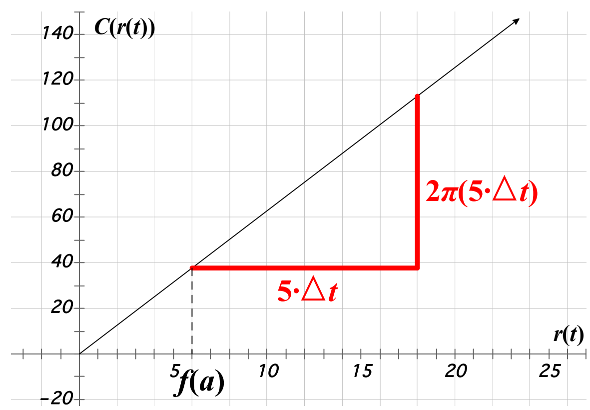
2. A rock is thrown into a lake, creating a circular ripple that travels outward at a rate of 5 inches per second.
   a. Define a function $r$ that expresses the radius of the circular ripple (in inches) in terms of $t$, the number of seconds elapsed since the rock hit the lake. $r(t) = 5t$.
   b. Define a function $C$ that expresses the circumference of the circular ripple (in inches) in terms of $r(t)$, the radius of the circular ripple. $C(r(t)) = 2\pi(r(t))$.
   c. Sketch a graph of the function $r$ with respect to $t$ on the left set of axes below. Sketch a graph of the function $C$ with respect to $r(t)$ on the right set of axes below.




   d. The axes on the left below show a change in $t$ from $t = a$ to $t = a + \Delta t$.



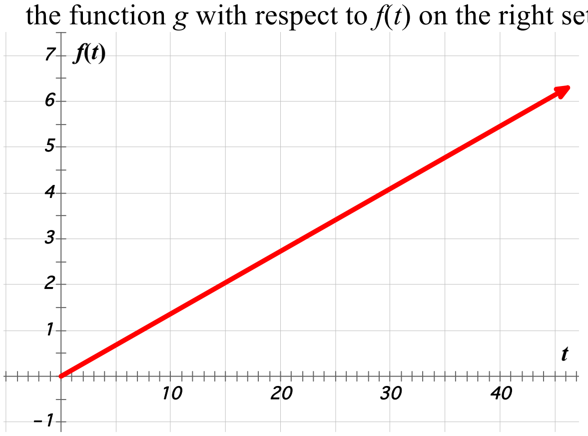

      i. On the left set of axes, illustrate the change in $r(t)$ that corresponds to a change in $t$ from $t = a$ to $t = a + \Delta t$ and represent this change symbolically.
      See the graph above. Make sure students recognize that since the radius of the circular ripple (in inches) varies at a constant rate of 5 with respect to the number of seconds elapsed since the rock hit the lake, the change in $r(t)$ that corresponds to a change in $t$ from $t = a$ to $t = a + \Delta t$ is given by $5 \cdot \Delta t$. That is, $\Delta r(t) = 5 \cdot \Delta t$.
      ii. On the right set of axes, illustrate the change in $r(t)$ that corresponds to a change in $t$ from $t = a$ to $t = a + \Delta t$ and represent this change symbolically.
      See the graph above. Make sure students recognize that a change in $r(t)$ of $5 \cdot \Delta t$ is represented as a horizontal change on the graph of $C(r(t))$ with respect to $r(t)$.
      iii. On the right set of axes, illustrate the change in $C(r(t))$ that corresponds to a change in $r(t)$ from $r(a)$ to $r(a + \Delta t)$ and represent this change symbolically.
      See the graph above. Make sure students recognize that since the circumference of the circular ripple (in inches) varies at a constant rate of $2\pi$ with respect to the radius of the circular ripple (in inches), the change in $C(r(t))$ that corresponds to a change in $r(t)$ from $r(a)$ to $r(a + \Delta t)$ is given by $2\pi \cdot \Delta r(t) = 2\pi(5 \cdot \Delta t)$.

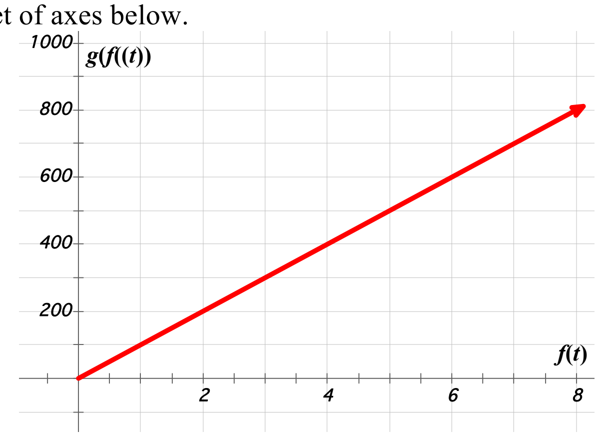
e. Refer to your graphs from Part (d) to write an expression that represents the rate of change of $C(r(t))$ with respect to $t$. Explain why the expression you wrote represents this rate of change.
The rate of change of $C(r(t))$ with respect to $t$ is given by

$$\frac{\Delta C(r(t))}{\Delta t} = \frac{2\pi(5 \cdot \Delta t)}{\Delta t} = 2\pi(5).$$

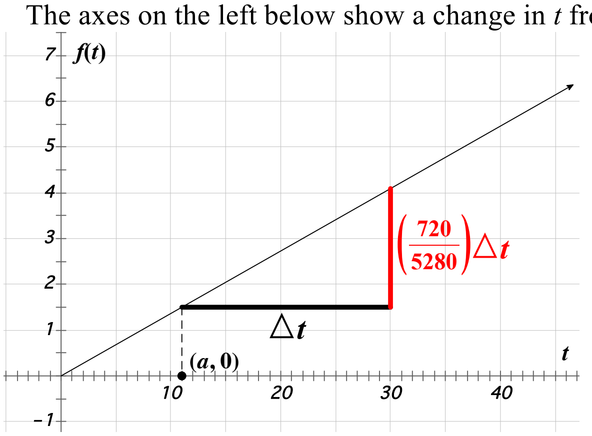
Before moving on to the next question, ask students to interpret what the values in the product $2\pi(5)$ represent in the context of this situation. We notice that this is the product of the constant rate of change of $r(t)$ with respect to $t$ and the constant rate of change of $C(r(t))$ with respect to $r(t)$. That the rate of change of the composite function $C$ is the product of these rates of change will be generalized in Problem 6. This, of course, is the conceptual foundation for the chain rule.

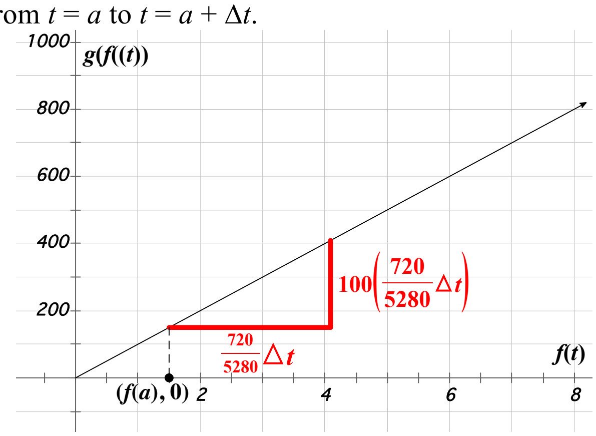
3. Suppose Courtney goes for a run, traveling at a constant speed of 720 feet per minute and burning 100 calories for every mile she runs.
   a. Define a function $f$ that expresses the distance Courtney has run (in miles) in terms of $t$, the number of minutes elapsed since Courtney started running. $f(t) = {}^{720t}/_{5280}$.
   b. Define a function $g$ that expresses the number of calories Courtney has burned since she started running in terms of $f(t)$, the distance Courtney has run (in feet). $g(f(t)) = 100 \cdot f(t)$.
   c. Sketch a graph of the function $f$ with respect to $t$ on the left set of axes below. Sketch a graph of the function $g$ with respect to $f(t)$ on the right set of axes below.




   d. The axes on the left below show a change in $t$ from $t = a$ to $t = a + \Delta t$.




      i. On the left set of axes, illustrate the change in $f(t)$ that corresponds to a change in $t$ from $t = a$ to $t = a + \Delta t$ and represent this change symbolically.
      See the graph above. Make sure students recognize that since the distance Courtney has run (in miles) varies at a constant rate of ${}^{720}/_{5280}$ with respect to the number of minutes she has

been running, the change in $f(t)$ that corresponds to a change in $t$ from $t = a$ to $t = a + \Delta t$ is given by $({}^{720}/_{5280})\Delta t$. That is, $\Delta f(t) = ({}^{720}/_{5280})\Delta t$.

ii. On the right set of axes, illustrate the change in $f(t)$ that corresponds to a change in $t$ from $t = a$ to $t = a + \Delta t$ and represent this change symbolically.
See the graph above. Make sure students recognize that a change in $f(t)$ of $({}^{720}/_{5280})\Delta t$ is represented as a horizontal change on the graph of $g(f(t))$ with respect to $f(t)$.

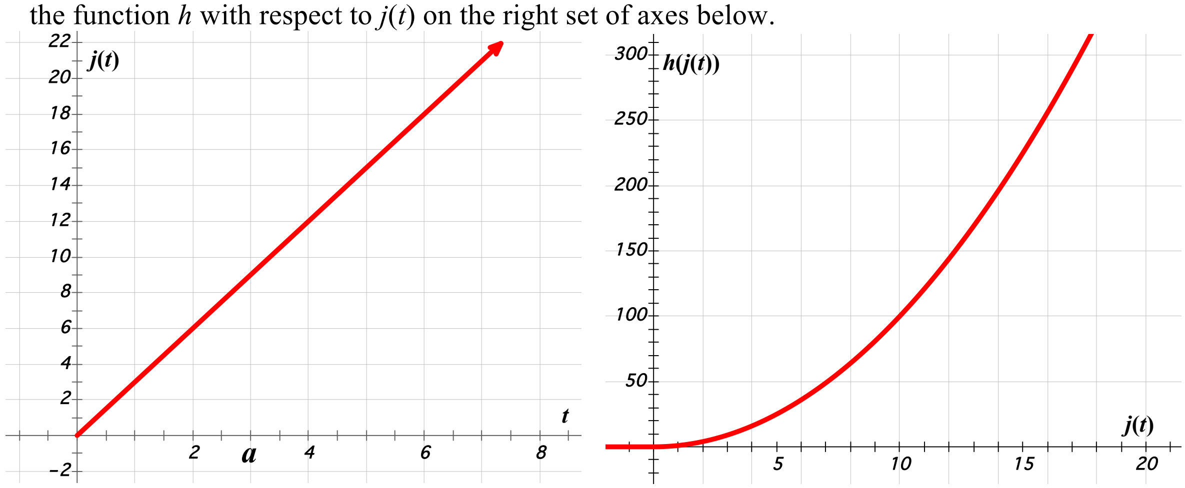
iii. On the right set of axes, illustrate the change in $g(f(t))$ that corresponds to a change in $f(t)$ from $f(a)$ to $f(a + \Delta t)$ and represent this change symbolically.
See the graph above. Make sure students recognize that since the number of calories Courtney has burned varies at a constant rate of 100 with respect to the number of miles she has run, the change in $g(f(t))$ that corresponds to a change in $f(t)$ from $f(a)$ to $f(a + \Delta t)$ is given by $100 \cdot \Delta f(t) = 100({}^{720}/_{5280})\Delta t$.

e. Refer to your graphs from Part (d) to write an expression that represents the rate of change of $g(f(t))$ with respect to $t$. Explain why the expression you wrote represents this rate of change.
The rate of change of $g(f(t))$ with respect to $t$ is given by

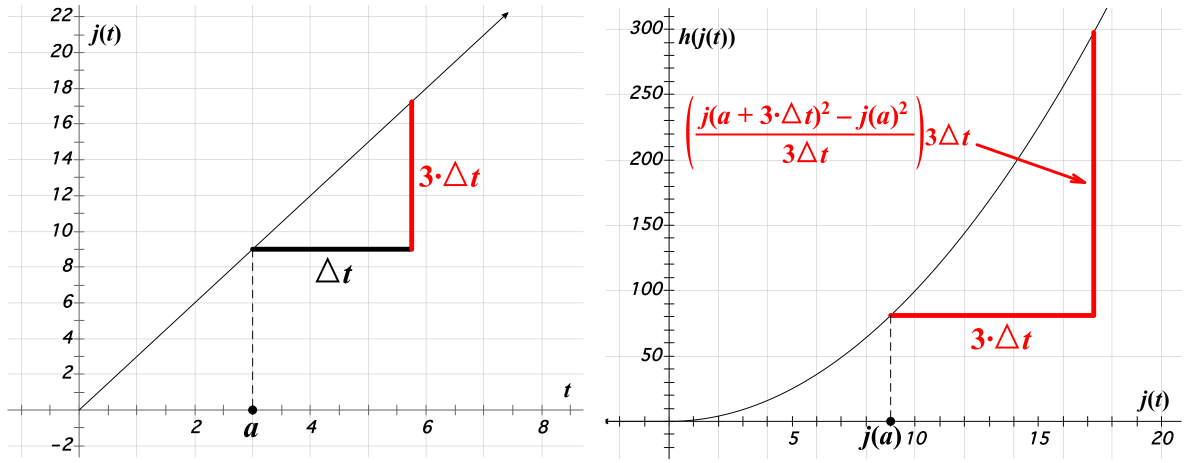
$$\frac{\Delta g(f(t))}{\Delta t} = \frac{100({}^{720}\!/_{5280})\Delta t}{\Delta t} = 100({}^{720}\!/_{5280}).$$

Before moving on to the next question, ask students to interpret what the values in the product $100({}^{720}/_{5280})$ represent in the context of this situation. We notice that this is the product of the constant rate of change of $f(t)$ with respect to $t$ and the constant rate of change of $g(f(t))$ with respect to $f(t)$. That the rate of change of the composite function $g$ is the product of these rates of change will be generalized in Problem 6. The purpose of this question, as with Part (e) of Problem 2, is to lay the conceptual foundation for the chain rule.

4. Imagine that a square is growing continuously so that the length of each side $s$ begins with a value of 0 inches and grows at a constant rate of 3 inches per second.
   a. Define a function $j$ that determines the side length of the square $s$ in terms of the number of seconds $t$ since the square started expanding from a side length of 0 inches. $j(t) = 3t$.
   b. Define a function $h$ that determines the area of the square (in square inches) in terms of $j(t)$, the side length of the square (in inches). $h(j(t)) = j(t)^2$.
   c. Sketch a graph of the function $j$ with respect to $t$ on the left set of axes below. Sketch a graph of the function $h$ with respect to $j(t)$ on the right set of axes below.




   d. The axes on the left below show a change in $t$ from $t = a$ to $t = a + \Delta t$.




i. On the left set of axes, illustrate the change in $j(t)$ that corresponds to a change in $t$ from $t = a$ to $t = a + \Delta t$ and represent this change symbolically.
See the graph above. Make sure students recognize that since the side length of the square (in inches) varies at a constant rate of 3 with respect to the number of seconds elapsed since the side length began increasing from a length of 0 inches, the change in $j(t)$ that corresponds to a change in $t$ from $t = a$ to $t = a + \Delta t$ is given by $3\cdot\Delta t$. That is, $\Delta j(t) = 3\cdot\Delta t$.

ii. On the right set of axes, illustrate the change in $j(t)$ that corresponds to a change in $t$ from $t = a$ to $t = a + \Delta t$ and represent this change symbolically.
See the graph above. Make sure students recognize that a change in $j(t)$ of $3\cdot\Delta t$ is represented as a horizontal change on the graph of $h(j(t))$ with respect to $j(t)$.

iii. On the right set of axes, illustrate the change in $h(j(t))$ that corresponds to a change in $j(t)$ from $j(a)$ to $j(a + \Delta t)$ and represent this change symbolically.
See the graph above. Although students are likely to represent the change in $h(j(t))$ as $j(a + 3\cdot\Delta t)^2 - j(a)^2$, it is important that before moving on you ask students represent the change in $h(j(t))$ as the product of the average rate of change of $h(j(t))$ with respect to $j(t)$ over the interval $[j(a), j(a + 3\cdot\Delta t)]$ and $3\cdot\Delta t$, the change in $j(t)$. Our rationale for doing so is to support students' recognition in Part (e) that the average rate of change of $h(j(t))$ with respect to $t$ is given by the product of the average rate of change of $h(j(t))$ with respect to $j(t)$ over the interval $[j(a), j(a + 3\cdot\Delta t)]$ and the constant rate of change of $j(t)$ with respect to $t$. Keep in mind that the purpose of Problems 2-4 is to allow students to abstract the structure of determining the rate of change of a composite function in a way that provides a conceptual foundation for the chain rule.

e. Refer to your graphs from Part (d) to write an expression that represents the rate of change of $h(j(t))$ with respect to $t$. Explain why the expression you wrote represents this rate of change.
The rate of change of $h(j(t))$ with respect to $t$ is given by

$$\frac{\Delta h(j(t))}{\Delta t} = \frac{\frac{j(a+3\Delta t)^2 - j(a)^2}{3\Delta t}\cdot 3\Delta t}{\Delta t} = \frac{j(a+3\Delta t)^2 - j(a)^2}{3\Delta t}\cdot 3.$$

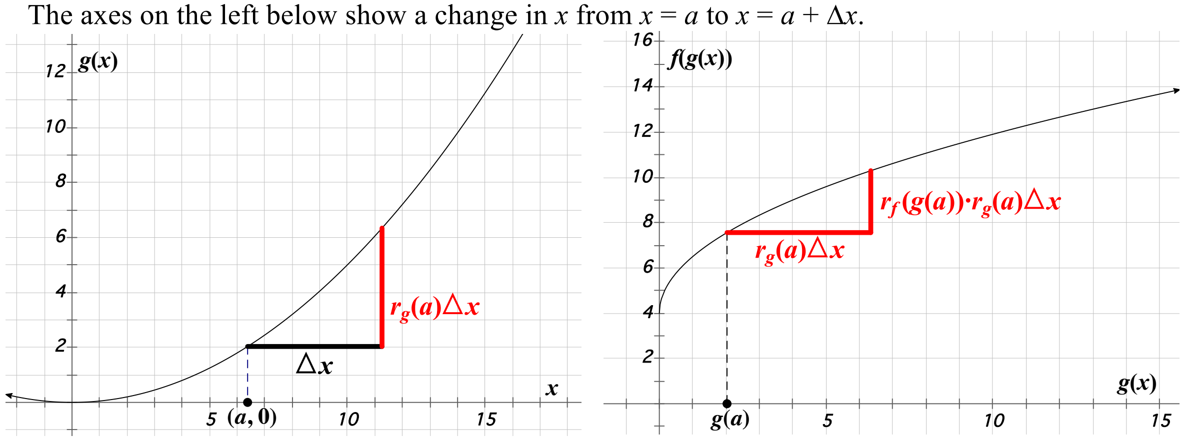
While the expression above can be further simplified, it is essential that you ask students to leave it in this form, and then to ask them to interpret what the values in the product $\frac{j(a+3\Delta t)^2 - j(a)^2}{3\Delta t}\cdot 3$ represent in the context of this situation. We notice that this is the product of the constant rate of change of $j(t)$ with respect to $t$ and the average rate of change of $h(j(t))$ with respect to $j(t)$ the interval $[j(a), j(a + 3\cdot\Delta t)]$. That the rate of change of the composite function $h$ is the product of these rates of change will be generalized in Problem 6. The purpose of this question, as with Part (e) of Problems 2 and 3, is to lay the conceptual foundation for the chain rule.

5. In Part (e) of Problems 2-4, we represented the average rate of change of composite functions with respect to their input variables. Review your work on these problems and make a conjecture about how to represent the average rate of change of the generic composite function $(f \circ g)(x)$ with respect to $x$. Feel free to state your conjecture using words.

The rate of change of $f \circ g(x)$ with respect to $x$ = The rate of change of $f(g(x))$ with respect to $g(x)$ X The rate of change of $g(x)$ with respect to $x$

6. We will now verify your conjecture from the previous task. First, we need to introduce some notation:
   - Let $r_g(a)$ represent the average rate of change of $g(x)$ with respect to $x$ from $x = a$ to $x = a + \Delta x$.
   - Let $r_f(g(a))$ represent the average rate of change of $f(g(x))$ with respect to $g(x)$ from $g(x) = g(a)$ to $g(x) = g(a + \Delta x)$.

   a. The axes on the left below show a change in $x$ from $x = a$ to $x = a + \Delta x$.




      i. On the left set of axes, illustrate the change in $g(x)$ that corresponds to a change in $x$ from $x = a$ to $x = a + \Delta x$ and represent this change symbolically.
      See the graph above. Make sure students recognize that the change in $g(x)$ that corresponds to a change in $x$ from $x = a$ to $x = a + \Delta x$ is the product of $\Delta x$ and the average rate of change of $g(x)$ with respect to $x$ over the interval $[a, a + \Delta x]$. That is, $\Delta g(x) = r_g(a)\Delta x$.

      ii. On the right set of axes, illustrate the change in $g(x)$ that corresponds to a change in $x$ from $x = a$ to $x = a + \Delta x$ and represent this change symbolically.
      See the graph above. Make sure students recognize that a change in $g(x)$ of $r_g(a)\Delta x$ is represented as a horizontal change on the graph of $f(g(x))$ with respect to $g(x)$.

      iii. On the right set of axes, illustrate the change in $f(g(x))$ that corresponds to a change in $g(x)$ from $g(a)$ to $g(a + \Delta x)$ and represent this change symbolically.
      See the graph above. Make sure students recognize that the change in $f(g(x))$ that corresponds to a change in $g(x)$ from $g(a)$ to $g(a + \Delta x)$ is the product of $r_g(a)\Delta x$ and the average rate of change of $f(g(x))$ with respect to $g(x)$ over the interval $[g(a), g(a + \Delta x)]$. That is, $\Delta f(g(x)) = r_f(g(a)) \cdot r_g(a)\Delta x$.

   b. Refer to your graphs from Part (d) to write an expression that represents the average rate of change of $f(g(x))$ with respect to $x$ over the interval $[a, a + \Delta x]$. Does the expression you wrote verify the conjecture you wrote in Task 5? Explain.

In the graph on the left above, we see that $\Delta x$ represents our initial change in $x$ and $r_g(a)\Delta x$ represents the change in $g(x)$ that corresponds to a change in $x$ from $x = a$ to $x = a + \Delta x$. Since we are interested in the rate of change of the composite function, the change in outputs of the graph on the left is shown as a change in inputs of the graph on the right. The change in output values of the graph on the right is given by the product of the rate of change of $f$ evaluated at the input $g(a)$ and the change in input values $r_g(a)\Delta x$. We see that the rate of change of $f(g(x))$ ***with respect to x*** is represented by the change in output values of the graph on the right divided by the change in input values of the graph on the left. Hence, we have

$$\frac{\Delta f(g(x))}{\Delta x} = \frac{r_f(g(a)) \cdot r_g(a)\Delta x}{\Delta x} = r_f(g(a)) \cdot r_g(a).$$

Since $r_f(g(a))$ represents the average rate of change of $f(g(x))$ with respect to $g(x)$ from $g(x) = g(a)$ to $g(x) = g(a + \Delta x)$, and since $r_g(a)$ represents the average rate of change of $g(x)$ with respect to $x$ from $x = a$ to $x = a + \Delta x$, the expression above validates the conjecture from Task 5.

As you noticed in Part (b) of Problem 6, the average rate of change of a generic composite function $(f \circ g)(x)$ with respect to $x$ is given by:

$$r_{f\circ g}(x) = r_f(g(x))r_g(x).$$

If we allow $\Delta x$ to approach zero, the average rates of change $r_f(g(x))$ and $r_g(x)$ respectively approach $f'(g(x))$ and $g'(x)$. Symbolically, as $\Delta x$ approaches zero we have

$$\lim_{\Delta x\to 0} r_{g\circ f}(x) = \lim_{\Delta x\to 0} r_f(g(x))r_g(x) = f'(g(x)) \cdot g'(x).$$

This observation suggests a method for computing the derivative of composite functions. We call this method the ***chain rule***.

***Chain Rule.*** Let $f$ and $g$ be differentiable at $x$. Then

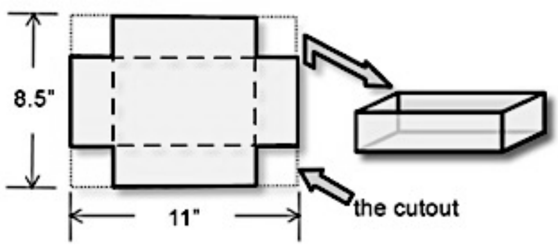
$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x).$$

Tasks 1-3 of this investigation support students' understanding that the critical points of a differentiable function occur at input values for which the derivative equals zero. Tasks 4-11 ask students to apply this understanding to determine the maximum or minimum value of some quantity, provided particular constraints. These problems vary in difficulty. Problems 7 and 11 are more computationally involved, and therefore prompt students to use their graphing calculators or a computer algebra system to determine the value of the input for which the derivative is equal to zero. This investigation does not include a homework section. It is unlikely that students will complete more than 2-3 of these problems in class, so feel free to select a few of the remaining problems for students to complete as homework.

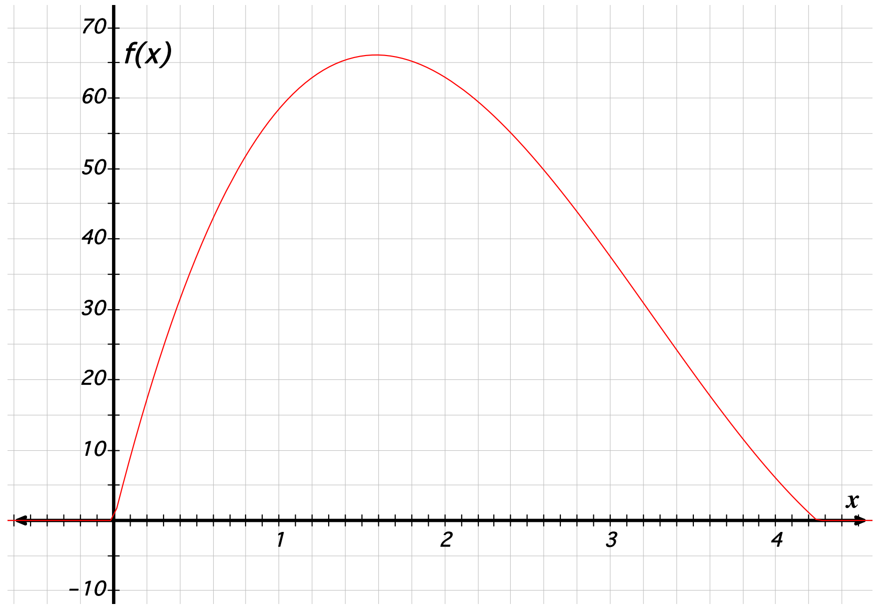
The following problems require you to determine the optimal value of some quantity. Consider the following while working each problem:

1. What quantities are changing in this problem? Assign variable names to represent the values of these varying quantities.
2. What quantities are fixed in the problem?
3. What quantity is being optimized?
4. Construct a formula that relates the quantity to be optimized to all of the other changing quantities. (This is the optimization formula.)
5. What constraints appear in the problem? (Focus on identifying inequalities and using the fixed quantities.)
6. Use the constraints to rewrite the optimization formula as a function of one variable.
7. Determine the relevant domain for the optimization function.
8. Solve the problem.

1. An open-top box can be created by cutting four equal-sized square corners from an 8.5 by 11-inch sheet of paper and folding up the sides (see image below).




   a. Define a function $f$ that determines the volume of the box (in cubic inches) provided the length of the side of the square cutout $x$ (in inches). Express the polynomial in both standard and factored form.
   Factored form: $f(x) = x(11 - 2x)(8.5 - 2x)$
   Standard form: $f(x) = 4x^3 - 39x^2 + 93.5x$
   b. Sketch a graph of the function $f$ over an appropriate domain.




c. What is the rate of change of $f(x)$ with respect to $x$ when the volume of the box is maximized? Explain.
Let $x = a$ be the input value where the maximum output for $f$ occurs. Consider input intervals determined by $x = a$ and $x = a + \Delta x$. In other words, consider the input intervals $[a, a + \Delta x]$ for positive $\Delta x$ and $[a + \Delta x, a]$ for negative $\Delta x$. As the values of $\Delta x$ get closer and closer to 0, the average rate of change for $f$ on the input intervals will be getting closer and closer to 0 cm$^3$/cm. The average rates of change taken over small input intervals where $\Delta x > 0$ will all be negative, while average rates of change taken over input intervals where $\Delta x > 0$ will all be positive because $f(a)$ is the largest output value on these intervals. Therefore, as $\Delta x$ approaches zero, the average rates of change of $f(x)$ with respect to $x$ approach zero, the value between positive and negative values.

d. Compute $f'(x)$.
$f'(x) = 12x^2 - 78 + 93.5$

e. Evaluate $f'(0.7)$ and explain what this value represents in the context of this situation.
$f'(0.7) = 12(0.7)^2 - 78(0.7) + 93.5 = 21.38$.
This means that the limiting value of the average rate of change of the volume of the box (in cubic centimeters) with respect to the length of the side of the cutout over the interval $[0.7, 0.7 + \Delta x]$ as $\Delta x$ approaches zero is 21.38 cm$^3$/cm. In other words, the local constant rate of change of $f(x)$ with respect to $x$ over a very small interval near $x = 0.7$ is 21.38 cm$^3$/cm.

f. Use your response to Parts (c) and (d) to determine the dimensions of the box with a maximum volume.
We can determine the value of $x$ for which $f(x)$, the volume of the box, is maximized by solving the equation $f'(x) = 0$.

$$0 = f'(x)$$

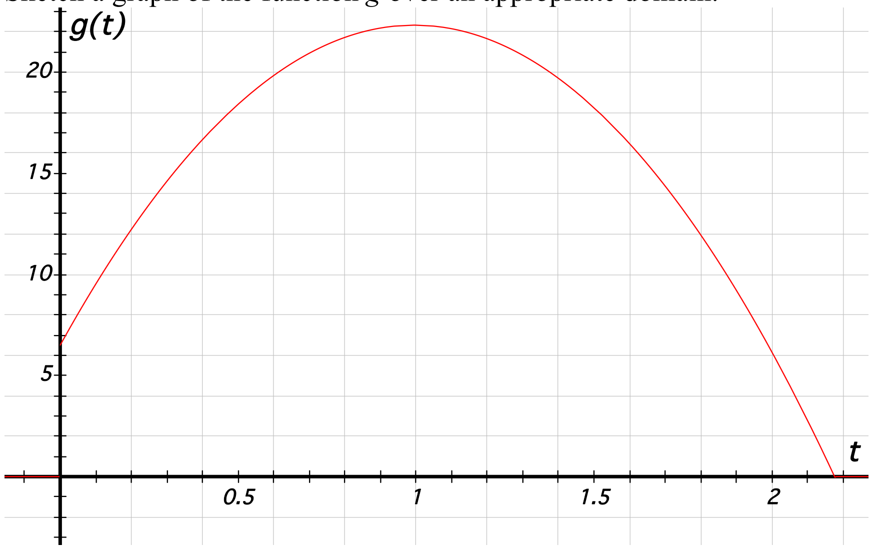
$$0 = 12x^2 - 78x + 93.5$$

$$x = \frac{78 \pm \sqrt{(-78)^2 - 4(12)(93.5)}}{2(12)}$$

$$x = 1.585, \quad x = 4.915$$

Since $x = 4.915$ is not in the practical domain of the function $f$, the value of $x$ for which $f(x)$ is maximized is $x = 1.585$. The maximum volume of the box is therefore $f(1.585) = 66.148$ cm$^3$.

2. Suppose a baseball outfielder fields a ball and throws it back towards the infield, releasing it from his hand 6.5 feet above ground level at an angle of 18º above the horizontal at a speed of 103 feet per second. Neglecting air resistance, the baseball's height above the ground $h$ (in feet) after $t$ seconds since it was released can be modeled by the function $g(t) = -16t^2 + 31.829t + 6.5$.
   a. Sketch a graph of the function $g$ over an appropriate domain.




   b. What is the rate of change of $g(t)$ with respect to $t$ when the ball's height above the ground is maximized? Explain.
   When the ball's height above the ground is maximized, the rate of change of $g(t)$ with respect to $t$ is zero feet per second. This is because there is no change in the value of $g(t)$ over a very small interval around the value of $t$ for which $g(t)$ achieves it maximum value. Therefore, the local constant rate of change of $g(t)$ with respect to $t$ when the ball's height above the ground is maximized is 0 ft/sec.
   c. Compute $g'(t)$.
   $g'(x) = -32t - 31.829$
   d. Evaluate $g'(1.25)$ and explain what this value represents in the context of this situation.
   $g'(1.25) = -32(1.25) - 31.829 = -71.83$.
   This means that the limiting value of the average rate of change of the ball's height above the ground (in feet) with respect to the number of seconds elapsed since the ball was released over the interval $[1.25, 1.25 + \Delta t]$ as $\Delta t$ approaches zero is $-71.83$ ft/sec. In other words, the local constant rate of change of $g(t)$ with respect to $t$ over a very small interval near $t = 1.25$ is $-71.83$ ft/sec.
   e. Use your responses to Parts (b) and (c) to determine the maximum height of the ball.
   We can determine the value of $t$ for which $g(t)$, the ball's height above the ground, is maximized by solving the equation $g'(t) = 0$.

$$0 = g'(t)$$

$$0 = -32t - 31.829$$

$$t \approx 0.995$$

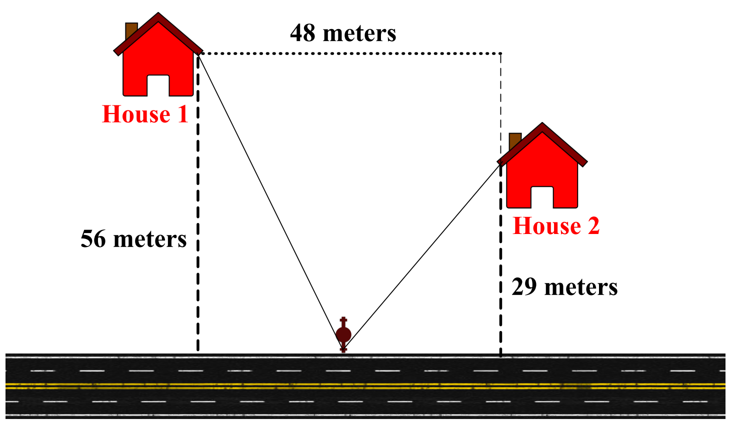
   The value of $t$ for which $g(t)$ is maximized is $t = 0.995$. The maximum height of the ball above the ground is therefore $g(0.995) = 22.33$ ft.

3. If the function $f$ is locally linear for all values $x$ in its domain and $f$ has a local maximum at $x = c$, then what must be true of $f'(c)$? Explain.
   Consider input intervals determined by $x = c$ and $x = c + \Delta x$. In other words, consider the input intervals $[c, c + \Delta x]$ for positive $\Delta x$ and $[c + \Delta x, c]$ for negative $\Delta x$. As the values of $\Delta x$ get closer and closer to 0, the average rate of change for $f$ on these input intervals will be getting closer and closer to zero. The average rates of change taken over small input intervals where $\Delta x > 0$ will all be negative, while average rates of change taken over input intervals where $\Delta x > 0$ will all be positive because $f(c)$ is the largest output value on these intervals. Therefore, as $\Delta x$ approaches zero, the average rates of change of $f(x)$ with respect to $x$ approach zero, the value between positive and negative values.

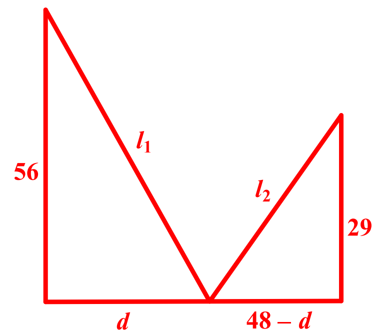
Problems 4–11 require you to determine the maximum or minimum value of some quantity, provided particular constraints. As you work each of these problems, keep in mind the following strategies:

- Conceptualize the situation, including the quantities involved. It is often very helpful to draw a picture (if one is not provided) and to label all relevant quantities on your picture.
- Identify the quantity whose value you need to maximize or minimize.
- Using the constraints in the problem, write a formula that expresses the quantity you need to maximize/minimize as a function of one other quantity.
- Determine the practical domain of the function you defined.

4. Two new houses need to be supplied with telephone lines. The telephone company needs to install a telephone pole immediately adjacent to a road that passes to the south of the two new houses (see the following image). The lines supplied to each house will extend directly from the telephone pole the company needs to install. One house is 56 meters from the road and the other is 29 meters from the road. The horizontal distance (east to west) between the two houses is 48 meters. The telephone company would like to install the pole in a location that minimizes the amount of wire needed to supply the two houses with telephone service.




We wish to minimize the sum of $l_1$ and $l_2$ in the diagram below.




Using the Pythagorean theorem, we have $l_1 = \sqrt{56^2 + d^2}$ and $l_2 = \sqrt{29^2 + (48-d)^2}$. We define the function $f$ to represent the sum of $l_1$ and $l_2$. We differentiate $f$, set equal to zero, and solve for $d$ to determine the critical points of $f$.

$$f(d) = \sqrt{56^2 + d^2} + \sqrt{29^2 + (48-d)^2}$$

$$f'(d) = \frac{d}{\sqrt{56^2 + d^2}} - \frac{48-d}{\sqrt{29^2 + (48-d)^2}}$$

$$0 = \frac{d}{\sqrt{56^2 + d^2}} - \frac{48-d}{\sqrt{29^2 + (48-d)^2}}$$

$$\frac{48-d}{\sqrt{29^2 + (48-d)^2}} = \frac{d}{\sqrt{56^2 + d^2}}$$

$$(48-d)\sqrt{56^2 + d^2} = d\sqrt{29^2 + (48-d)^2}$$

$$(48-d)^2(56^2 + d^2) = d^2(29^2 + (48-d)^2)$$

$$56^2(48-d)^2 + d^2(48-d)^2 = 29^2 d^2 + d^2(48-d)^2$$

$$56^2(48-d)^2 = 29^2 d^2$$

$$56^2(48^2) - 56^2(96d) + 56^2 d^2 = 29^2 d^2$$

$$0 = (56^2 - 29^2)d^2 - 56^2(96d) + 56^2(48^2)$$

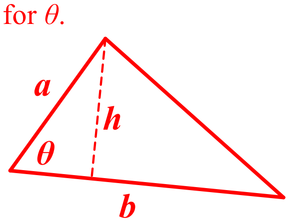
Using the quadratic formula, we find that the solutions to the equation above are $d \approx 99.556$ and $d \approx 31.624$. We find that $f'(31.624) = 0$ but $f'(99.556) \neq 0$. Therefore, $d \approx 99.556$ is an extraneous solution. Since $f$ is continuous, $f'(a) < 0$ for $a < 31.624$, and $f'(a) > 0$ for $a > 31.624$, we know that $f$ has an absolute minimum at $d = 31.624$. Therefore, the phone company should install the telephone pole 31.624 meters to the east of House 1, or 48 – 31.624 meters to the west of House 2.

5. Determine the dimensions of a cylindrical can that uses the least amount of metal and has a volume of 356 cubic centimeters.
   The volume $V$ and surface area $A$ of a cylinder are respectively given by $V = \pi r^2 h$ and $A = 2\pi rh + 2\pi r^2$, where $r$ is the radius of the cylinder and $h$ is its height. We seek to minimize $A$ provided the volume of the cylinder is 356 cm$^3$. We need to express the surface area of the can in terms of one variable, so we solve the equation $356 = \pi r^2 h$ for $h$ to get $h = 356/(\pi r^2)$ (we could just as well solve for $r$). We can now express $V$ as a function of $r$, differentiate, set equal to zero, and solve for $r$.

$$V = f(r) = 2\pi r\left(\frac{356}{\pi r^2}\right) + 2\pi r^2$$
$$f(r) = \frac{712}{r} + 2\pi r^2$$
$$f'(r) = -\frac{712}{r^2} + 4\pi r$$
$$0 = -\frac{712}{r^2} + 4\pi r$$
$$\frac{712}{r^2} = 4\pi r$$
$$178 = \pi r^3$$
$$r = \sqrt[3]{\frac{178}{\pi}}$$

Since $f$ is continuous for $r > 0$, $f'(a) < 0$ for $0 < a < \sqrt[3]{\frac{178}{\pi}}$, and $f'(a) > 0$ for $a > \sqrt[3]{\frac{178}{\pi}}$, we know that $f$ has at least a local minimum at $r = \sqrt[3]{\frac{178}{\pi}}$. Indeed, $f$ has an absolute minimum at $r = \sqrt[3]{\frac{178}{\pi}}$ if we restrict the domain of $f$ to only non-negative values. Therefore, dimensions of the can with a volume of minimize 356 cm$^3$ and minimum surface area are $r = \sqrt[3]{\frac{178}{\pi}}$ and $h = \dfrac{356}{\pi\left(\frac{178}{\pi}\right)^{2/3}}$.

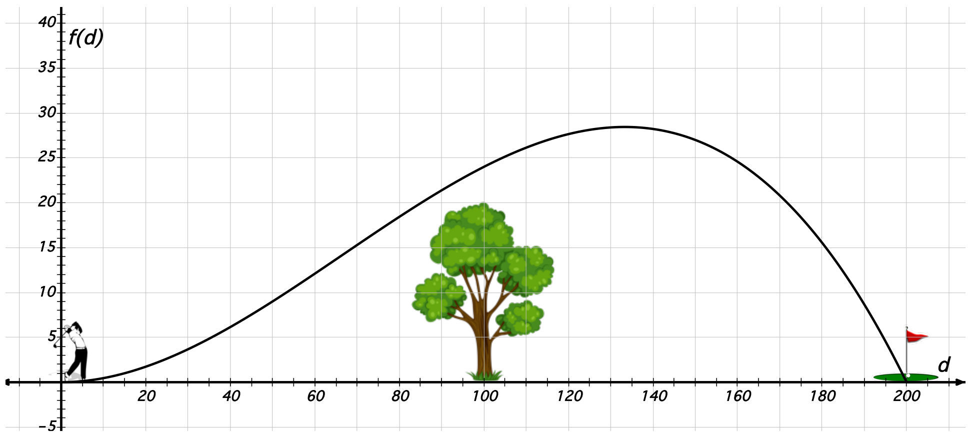
6. One side of a triangle has length $a$ and the other side has length $b$. Determine the length of the third side so that the area of the triangle is maximized.
The area $A$ of the triangle is given by $A = ½\cdot bh$. We express this area as a function of $\theta$, the measure of the angle created by the sides of lengths $a$ and $b$. We then differentiate, set equal to zero, and solve for $\theta$.




$$h = a\sin(\theta)$$
$$A = g(\theta) = \tfrac{1}{2}ab\sin(\theta)$$
$$g'(\theta) = \tfrac{1}{2}ab\cos(\theta)$$
$$0 = \tfrac{1}{2}ab\cos(\theta)$$
$$0 = \cos(\theta)$$
$$\theta = \frac{\pi}{2} + k\pi,\quad k \in \mathbb{Z}$$

The only solution to $g'(\theta) = 0$ in the practical domain of $g$ is $\pi/2$. The practical domain of the function $g$ is $[0, \pi]$. Since $(\pi/2, g(\pi/2))$ is the only critical point in the practical domain of the continuous function $g$, and since $g(0) = g(\pi) = 0$, we know that $g$ achieves its maximum value when $\theta = \pi/2$. Hence, the area of the triangle is maximized when it is a right triangle. By the Pythagorean theorem, the third side of the triangle is $\sqrt{a^2 + b^2}$.

7. Kevin was playing a weekend golf match against Michael. On the eighteenth hole Kevin found his ball exactly 200 yards from the hole. Unfortunately, there was a 20-yard tall tree halfway between Kevin's ball and the hole. In a stroke of brilliance, Kevin hit his ball over the tree and landed it inches away from the hole! (See the image below.) The function $f(d) = 0.0048d^2 - 0.000024d^3$ represents the relationship between the ball's height above the ground $f(d)$ (in yards) in terms of the ball's horizontal distance $d$ from Kevin (in yards). How close was Kevin's ball to the top of the tree? (*Define an appropriate function and compute its derivative by hand, but then use a computer algebra system or a graphing calculator to determine the value of the input for which the derivative equals zero.*)




We seek to minimize the distance $D$ between points on the graph of $f$ and the point (100, 20). Using the distance formula, we obtain,

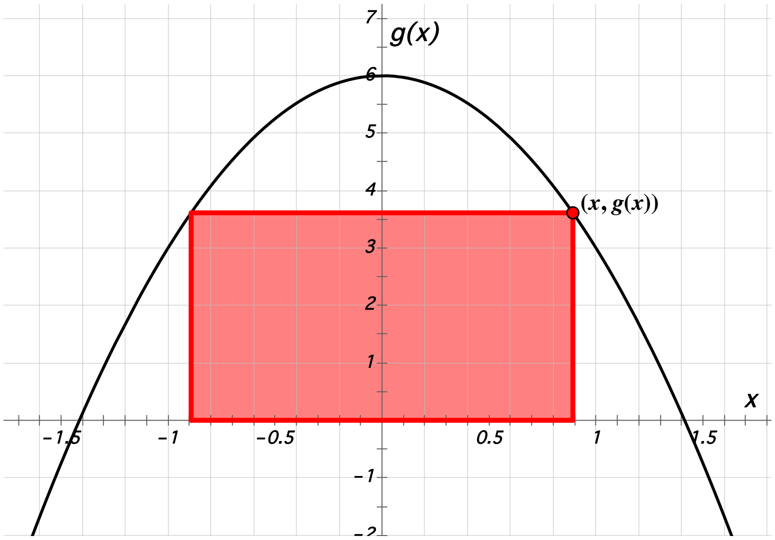
$$D = g(d) = \sqrt{(100-d)^2 + (20 - f(d))^2} = \sqrt{(100-d)^2 + \left(20 - (0.0048d^2 - 0.000024d^3)\right)^2}\ .$$

Our task now is to minimize $g(d)$, so we differentiate $g$.

$$g'(d) = \frac{-2(100-d) + 2(20 - 0.0048d^2 + 0.000024d^3)(-0.0096d + 0.000072d^2)}{\sqrt{(100-d)^2 + \left(20 - (0.0048d^2 - 0.000024d^3)\right)^2}}\ .$$

Using a computer algebra system (wolframalpha.com in this case), we solve $g'(d) = 0$ and obtain the real solution $d = 99.0771$. It is clear from the context that $g$ has an absolute minimum when $d = 99.0771$. Therefore, at its closest Kevin's ball was $g(99.0771) = 3.88761$ meters from the top of the tree.

8. A rectangle with its sides parallel to the coordinate axes is inscribed in the region enclosed by the graph of $g(x) = -3x^2 + 6$ and the $x$-axis. Find the dimensions of the rectangle with maximum area.
If we let the point $(x, g(x))$ represent the coordinates of the vertex of the rectangle in the first quadrant, then the area of the rectangle is given by $A = 2x(g(x)) = 2x(-3x^2 + 6) = -18x^3 + 12x$. We compute ${}^{dA}/_{dx}$ and set equal to zero to identify the critical points.



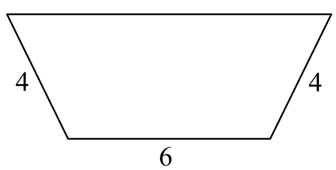

$$\frac{dA}{dx} = -18x^2 + 12$$

$$0 = -18x^2 + 12$$

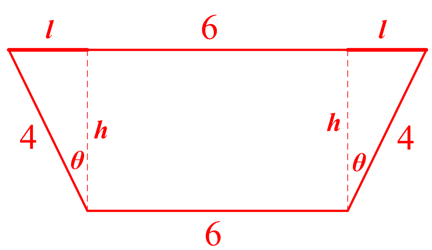
$$x = \pm\sqrt{\frac{2}{3}}$$

Although it appears that there are two solutions, there is only one since negative solution corresponds to the vertex of the same rectangle in the second quadrant. The rectangle with maximum area has a base of $2\sqrt{\frac{2}{3}}$ and a height of $-3\left(\sqrt{\frac{2}{3}}\right)^2 + 6 = 4$.

9. Find the measures of the interior angles that maximize the area of an isosceles trapezoid with a base of length 6 and sides of length 4.




We see that the area $A$ of the trapezoid below is $A = ½(12 + 2l)h = (6 + l)h$. We express $A$ as a function of $h$ by letting $l = 4\sin(\theta)$ and $h = 4\cos(\theta)$. We compute ${}^{dA}/_{d\theta}$ and set equal to zero to determine the value of $\theta$ for which the area of the trapezoid is maximized.



$$A=(6+4\sin(\theta))\cdot 4\cos(\theta)$$

$$\begin{aligned}\frac{dA}{d\theta}&=16\cos^2(\theta)-4\sin(\theta)(6+4\sin(\theta))\\&=16\cos^2(\theta)-16\sin^2(\theta)-24\sin(\theta)\\&=16(1-\sin^2(\theta))-16\sin^2(\theta)-24\sin(\theta)\\&=-32\sin^2(\theta)-24\sin(\theta)+16\end{aligned}$$

$$0=32\sin^2(\theta)+24\sin(\theta)-16$$

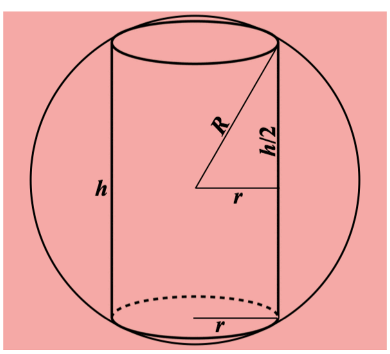
Since $\frac{dA}{d\theta}$ is a quadratic with respect to $\sin(\theta)$, we apply the quadratic formula.

$$\sin(\theta)=\frac{-24\pm\sqrt{24^2-4(32)(-16)}}{2(32)}$$

$$\theta=\arcsin\left(\frac{-24+\sqrt{24^2-4(32)(-16)}}{2(32)}\right)\approx 0.4394$$

Note that we consider only the positive square root since arcsine is undefined for the negative square root. It is clear from the context that the area of the trapezoid is maximized when $\theta$ is approximately 0.4394 radians. Therefore, the obtuse interior angles of the trapezoid with maximum area are ($0.4394 + \pi/2$) radians and the acute angles are ($\pi/2 - 0.4394$) radians.

10. A right cylinder is inscribed in a sphere of fixed radius $R$. Let $r$ and $h$ represent the radius and height of the cylinder respectively. Determine the dimensions of the cylinder that maximize its volume.
The volume $V$ of a right cylinder is given by $V = \pi r^2 h$. We use the fact that $R^2 = ({}^h/_2)^2 + r^2$ to express $V$ as a function of one variable (see the image below). We then differentiate $V = f(r)$ and set equal to zero to determine the value of $r$ for which the volume of the cylinder is maximized.




$$R^2=\left(\tfrac{h}{2}\right)^2+r^2 \quad\rightarrow\quad h=2\sqrt{R^2-r^2}$$

$$V=f(r)=\pi r^2\left(2\sqrt{R^2-r^2}\right)=2\pi\left(r^2\sqrt{R^2-r^2}\right)$$

$$\begin{aligned}f'(r)&=2\pi\left(2r\sqrt{R^2-r^2}-\frac{r^3}{\sqrt{R^2-r^2}}\right)\\&=2\pi\left(\frac{2R^2r-3r^3}{\sqrt{R^2-r^2}}\right)\\&=\frac{4\pi R^2r-6\pi r^3}{\sqrt{R^2-r^2}}\end{aligned}$$

$$0=\frac{4\pi R^2r-6\pi r^3}{\sqrt{R^2-r^2}}$$

$$0=4\pi R^2r-6\pi r^3$$

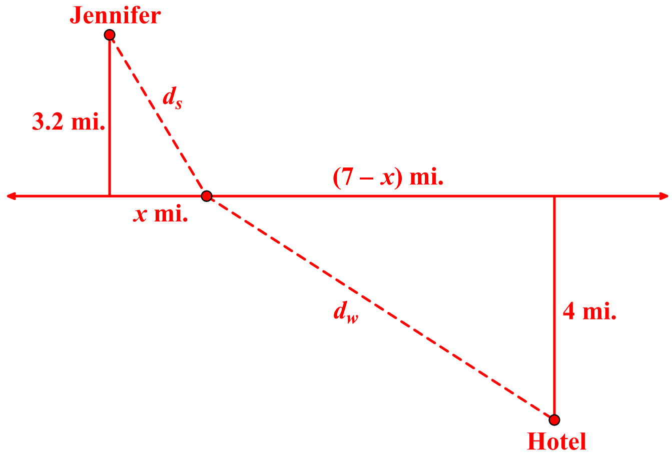
$$0=2R^2-3r^2$$

$$r=\sqrt{\frac{2R^2}{3}}$$

The dimensions of the right cylinder with maximum volume are $r=\sqrt{\frac{2R^2}{3}}$ and $h=2\sqrt{R^2-\frac{2R^2}{3}}$.

11. Jennifer is on vacation in Jamaica. While swimming in the Caribbean Sea, she finds herself located 3.2 miles from the closest point on a straight shoreline. She needs to reach her hotel located 7 miles down shore from the closest point and 4 mi inland. If she swims at 3.9 mi/hr and she walks at 5.4 mi/hr, how far from her hotel should she come ashore so that she arrives at her hotel in the shortest amount of time? (*Define an appropriate function and compute its derivative by hand, but then use a computer algebra system or a graphing calculator to determine the value of the input for which the derivative equals zero.*)

Refer to the diagram below. Let $d_s$ represent the distance Jennifer swims and let $d_w$ represent the distance Jennifer walks.




We can express $d_s$ and $d_w$ in terms of $x$.

$$d_s=\sqrt{x^2+3.2^2}$$
$$d_w=\sqrt{(7-x)^2+4^2}$$

Let $t_s$ represent the amount of time (in hours) it takes Jennifer to swim and let $t_w$ represent the amount of time (in hours) it takes Jennifer to walk. Let $T$ represent the total amount of time (in hours) it takes Jennifer to reach her hotel. Then $T = t_s + t_w$. We can express $T$ as a function of $x$ and then differentiate with respect to $x$ to determine where on the shoreline Jennifer should swim to minimize the amount of time it takes for her to reach her hotel.

$$T=t_s+t_w=\frac{d_s}{3.9}+\frac{d_w}{5.4}$$
$$T=f(x)=\frac{\sqrt{x^2+3.2^2}}{3.9}+\frac{\sqrt{(7-x)^2+4^2}}{5.4}$$
$$f'(x)=\frac{x}{3.9\sqrt{x^2+3.2^2}}+\frac{x-7}{5.4\sqrt{(7-x)^2+4^2}}$$

Using a computer algebra system (wolframalpha.com in this case), we solve $f'(x) = 0$ and obtain the real solution $d = 2.1477$. It is clear from the context that $f$ has an absolute minimum when $x = 2.1477$. Therefore, to minimize the amount of time it will take Jennifer to reach her hotel, she should swim 2.1477 miles in the direction of the hotel from the closest point on the shoreline. In other words, Jennifer should come ashore $7 - 2.1477 = 4.8523$ miles from her hotel.

This investigation begins by introducing students to the notion of a related rate formula. Problems 1–3 ask students to define formulas that express the relationship between two rates of change. The key idea is for students to recognize that if they define a formula that relates the values of two quantities, they can differentiate both sides with respect to the same independent variable to obtain a related rate formula. Make sure students understand why it is important to apply the chain rule when differentiating both sides of an equation with respect to some other variable (often time). Problems 4–8 require students to define a related rate formula and to solve it to determine the rate of change of one quantity with respect to another. Encourage students to take the time to carefully conceptualize each situation. These problems vary in difficulty. It is unlikely that students will complete more than one or two of these problems in one class session. Feel free to assign the remaining problems, or a subset of them, for homework.

1. Imagine a square that is increasing in size. Let $t$ represent the number of seconds elapsed since the square began increasing in size. Write a formula that defines the relationship between the rate at which the square's area $A$ is changing with respect to $t$ and the rate at which the square's side length $s$ is changing with respect to $t$.

$$A = s^2$$

$$\frac{d}{dt}(A) = \frac{d}{dt}(s^2)$$

$$\frac{dA}{dt} = 2s \cdot \frac{ds}{dt}$$

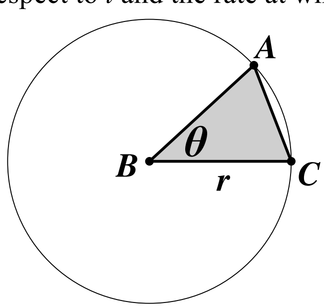
2. Imagine that a sphere is increasing in size. Let $t$ represent the number of seconds elapsed since the sphere began increasing in size. Write a formula that defines the relationship between the rate at which the sphere's volume $V$ is changing with respect to $t$ and the rate at which the sphere's radius $r$ is changing with respect to $t$.

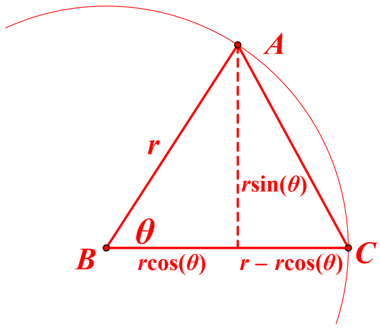
$$V = \frac{4}{3}\pi r^3$$

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

$$\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$$

3. Suppose $\theta$, the measure of $\angle ABC$ in the figure below, varies from 0 radians to $\pi/2$ radians. Let $t$ represent the number of seconds elapsed since $\theta$ started varying and let $r$ represent the fixed radius of the circle centered at $B$. Write a formula that defines the relationship between the rate at which the area $A$ of $\Delta ABC$ is changing with respect to $t$ and the rate at which $\theta$ is changing with respect to $t$.






$$A = \frac{r\cos(\theta)\cdot r\sin(\theta)}{2} + \frac{(r - r\cos(\theta))\cdot r\sin(\theta)}{2}$$
$$A = \frac{r^2\sin(\theta)}{2}$$
$$\frac{d}{dt}(A) = \frac{d}{dt}\left(\frac{r^2\sin(\theta)}{2}\right)$$
$$\frac{dA}{dt} = \frac{r^2\cos(\theta)}{2}\cdot\frac{d\theta}{dt}$$

The formulas you defined in Problems 1-3 are called ***related rate formulas*** since they define the relationship between two (or more) rates of change. Therefore, when you see the term, "related rates" in this course, you should think, "equation that defines the relationship between rates of change, or derivatives."

Let's examine the concept of related rate formulas in general. Let $A$ represent the measure of Quantity A and let $B$ represent the measure of Quantity B. Suppose the function $f$ defines the relationship between $A$ and $B$ so that $A = f(B)$. Further suppose that $A$ and $B$ are both functions of $x$, the measure of some other quantity. Then the related rate formula that defines the relationship between the rate of change of $A$ with respect to $x$ and the rate of change of $B$ with respect to $x$ is given by

$$\frac{dA}{dx} = \frac{df}{dB}\cdot\frac{dB}{dx}.$$

We define related rate formulas by first defining a formula that the values of the quantities whose rates of change we seek to relate. We can then differentiate both sides of the formula to change our standard formula into a related rate formula. You'll notice that the related rate formula above resembles the chain rule. This makes sense because the function $f$ is a composite function with respect to $x$. When defining a related rate formula, we therefore need to *remember to apply the chain rule*.

Problems 4–8 require you to define related rate formulas and then solve them to determine the rate of change of one quantity with respect to another. As with optimization problems, it is important that you take time to develop an image of the quantities involved in the situation. Then you are able to write an appropriate formula that defines the relationship between relevant quantities.

4. Water is being poured into a hemispherical bowl of radius 8 inches at the constant rate of 5 in$^3$/sec. Let $V$ represent the volume of water in the hemispherical bowl (in cubic inches) and let $h$ represent the height of water in the bowl (in inches). Then $V = \pi h^2(R - {}^{h}/_{3})$, where $R$ represents the radius of the bowl in inches.
   a. Determine the rate at which the water level is rising at the moment the water is 2.4 inches deep.
      We need to determine ${}^{dh}/_{dt}$ when $h = 2.4$. We are given that ${}^{dV}/_{dt} = 5$.

$$V = \pi h^2 (R - h/3) = \pi R h^2 - \frac{\pi h^3}{3}$$

$$\frac{dV}{dt} = 2\pi R h \cdot \frac{dh}{dt} - \pi h^2 \cdot \frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{dh}{dt}\left(2\pi R h - \pi h^2\right)$$

$$\frac{dh}{dt} = \frac{dV}{dt} \cdot \frac{1}{2\pi R h - \pi h^2}$$

$$\frac{dh}{dt} = \frac{5}{2\pi(8)(2.4) - \pi(2.4)^2} \approx 0.049$$

Therefore, the water level is rising at a rate of approximately 0.049 in/sec when the water is 2.4 inches deep.

b. What is the rate at which the circular surface of the water $A$ is changing when the water is 2.4 inches deep?

We need to determine ${}^{dA}/_{dt}$ when $h = 2.4$. We are given that ${}^{dV}/_{dt} = 5$. From Part (a), we know that ${}^{dh}/_{dt} \approx 0.049$. Let $r$ represent the radius of the circular surface area of the water. Using coordinate geometry, we find that $r = \sqrt{8^2 - (8-h)^2}$ .

$$A = \pi\left(\sqrt{8^2 - (8-h)^2}\right)^2 = \pi\left(8^2 - (8-h)^2\right) = 16h - \pi h^2$$

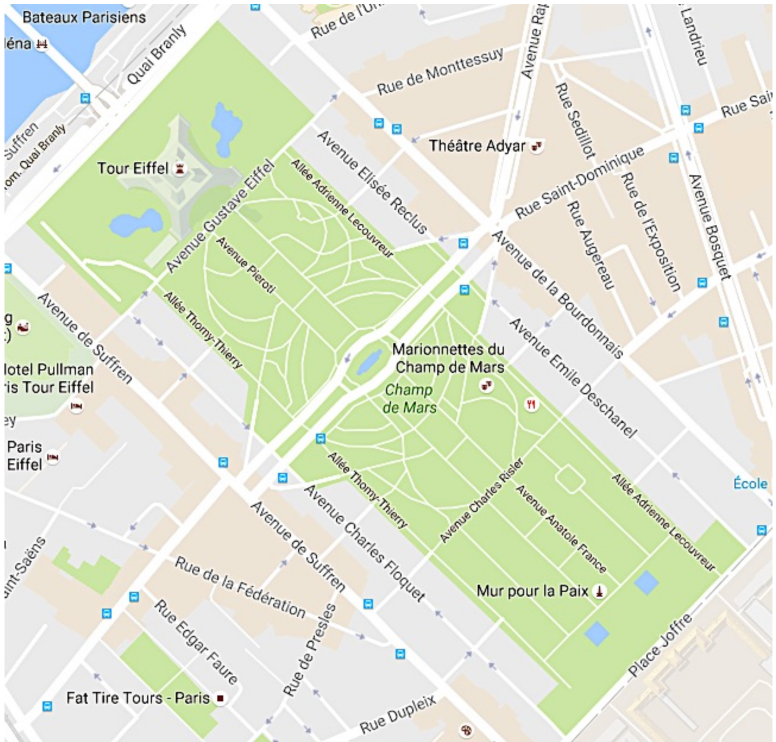
$$\frac{dA}{dt} = 16\pi \cdot \frac{dh}{dt} - 2\pi h \cdot \frac{dh}{dt}$$

$$\frac{dA}{dt} \approx 16\pi(0.049) - 2\pi(2.4)(0.049)$$

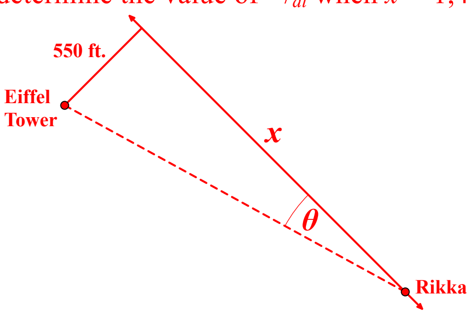
$$\frac{dA}{dt} \approx 1.72$$

Therefore, the circular surface of the water is increasing at a rate of approximately 1.72 $\text{in}^2$/sec when the water is 2.4 inches deep.

5. Rikka is driving along Avenue de la Bourdonnais in Paris at a constant rate of 44 feet per second. The Avenue de la Bourdonnais is a straight road adjacent to the Champ de Mars (a large rectangular greenspace in front of the Eiffel Tower). The road passes 550 feet to the right of the Eiffel Tower at its closest point (see the image below from https://maps.google.com). At what rate is the (acute) angle between the Avenue de la Bourdonnais and Rikka's sightline to the Eiffel Tower changing when she is 1,423 feet from the closest point on the Avenue de la Bourdonnais to the Eiffel Tower?




As indicated in the diagram below, we let $x$ represent the distance between Rikka and the point on the Avenue de la Bourdonnais closest to the Eiffel Tower. We also let $\theta$ represent the acute angle between the Avenue de la Bourdonnais and Rikka's sightline to the Eiffel Tower. We seek to determine the value of $^{d\theta}/_{dt}$ when $x = 1{,}423$ provided $^{dx}/_{dt} = -44$.




$$\tan(\theta) = \frac{550}{x} \quad \rightarrow \quad \theta = \arctan\left(\frac{550}{x}\right)$$

$$\frac{d}{dt}\big(\tan(\theta)\big) = \frac{d}{dt}\left(\frac{550}{x}\right)$$

$$\sec^2(\theta) \cdot \frac{d\theta}{dt} = \frac{-550}{x^2} \cdot \frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{\sec^2(\theta)}\left(\frac{-550}{x^2} \cdot \frac{dx}{dt}\right)$$

$$\frac{d\theta}{dt} = \frac{1}{\sec^2\left(\arctan\left(\frac{550}{1423}\right)\right)}\left(\frac{-550}{1423^2}(-44)\right) \approx 0.0104$$

Therefore, the (acute) angle between the Avenue de la Bourdonnais and Rikka's sightline to the Eiffel Tower changing at a rate of 0.0104 radians per second when she is 1,423 feet from the closest point on the Avenue de la Bourdonnais to the Eiffel Tower.

6. Imagine a point $P$ sliding along the graph of the function $f(x) = {}^1/_x$ from left to right in the first quadrant. The $x$-coordinate of point $P$ is increasing at a rate of 1.6 units per second. A right triangle is enclosed by the line tangent to the graph of $f$ at point $P$ and the positive $x$ and $y$ axes. Determine the

rate at which the $x$-intercept of the line tangent to $P$ is changing when the length of the triangle's hypotenuse is minimized.

If students are attending to the meaning of partial results while solving this task, they should recognize that this problem is simpler than it first appears. Some students will attempt to determine the location of point $P$ for which the hypotenuse of the triangle is minimized. This is unnecessary. Use this task as an opportunity to discuss the importance of being reflecting while solving novel problems.

Let $L$ represent the linear function tangent to the graph of $f$ at point $P$. Also, let $x_p$ represent the $x$-coordinate of point $P$.

$$f(x) = \frac{1}{x}$$

$$f'(x) = -\frac{1}{x^2}$$

$$L(x) = -\frac{1}{x_p{}^2}(x - x_p) + \frac{1}{x_p}$$

The $x$-intercept of $L$ is given by

$$0 = -\frac{1}{x_p{}^2}(x - x_p) + \frac{1}{x_p}$$

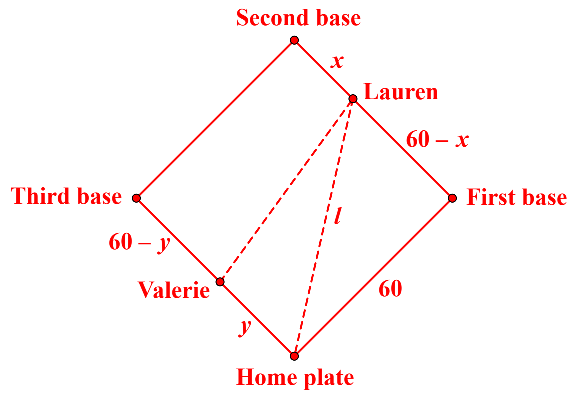
$$0 = -(x - x_p) + x_p$$

$$x = 2x_p$$

Consider the equation $\overline{x} = 2x_p$. So that we don't to confuse the $x$-intercept of the line tangent to the graph of $f$ with the input variable $x$, we let $\overline{x}$ represent this $x$-intercept. We notice that $\frac{d\overline{x}}{dt} = 2 \cdot \frac{dx_p}{dt}$.

Therefore, since $\frac{dx_p}{dt} = 1.6$, we have $\frac{d\overline{x}}{dt} = 3.2$. This means that the $x$-intercept of the line tangent to $P$ is increasing at a rate of 3.2 units per second at all moments in time while $P$ is sliding along the graph of $f$.

7. A softball diamond is a square with each side 60 feet in length. At the exact moment the ball is hit, Lauren runs from first base to second base at a constant rate of 17.6 feet per second and Valerie runs from third base to home plate at a constant rate of 15.5 feet per second.
   Let $x$ represent Lauren's distance from second base (in feet), let $l$ represent Lauren's distance from home plate (in feet), and let $y$ represent Valerie's distance from home plate (in feet) as shown in the figure below.




a. At what rate is Lauren's distance from home base changing when her distance from second base is 21.8 feet?
We would like to determine $^{dl}/_{dt}$ when $x = 21.8$. We are given $^{dx}/_{dt} = -17.6$. Using the Pythagorean theorem, we express $l$ as a function of $x$.

$$l=\sqrt{60^2+(60-x)^2}$$

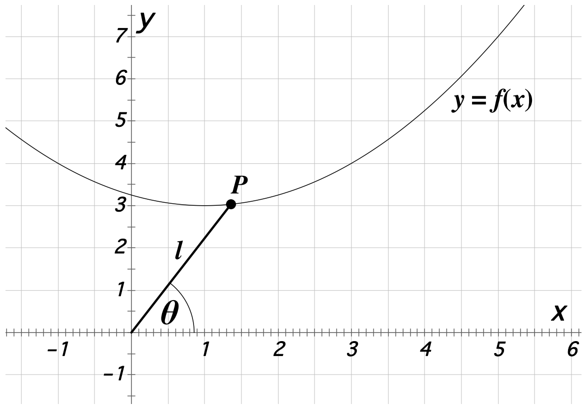
$$\frac{dl}{dt}=\frac{x-60}{\sqrt{60^2+(60-x)^2}}\cdot\frac{dx}{dt}$$

$$\frac{dl}{dt}=\frac{21.8-60}{\sqrt{60^2+(60-21.8)^2}}(-17.6)\approx 9.45$$

Therefore, Lauren's distance from home base increases at a rate of approximately 9.45 ft/sec when her distance from second base is 21.8 feet.

b. What is the constant rate at which the area of the triangle formed by connecting Lauren's position, Valerie's position, and third base changing as Lauren runs from first base to second base?
The area $A$ of the triangle formed by connecting Lauren's position, Valerie's position, and third base is given by $A=\frac{60y}{2}=30y$. Therefore, $\frac{dA}{dt}=30\cdot\frac{dy}{dt}$. Since Valerie is running toward home plate at a constant rate of 15.5 ft/sec, we have $\frac{dA}{dt}=30(-15.5)=-465$. This means that the area of the triangle by connecting Lauren's position, Valerie's position, and third base is decreasing at a constant rate of 465 $ft^2$/sec while Lauren is running from first base to second base.

8. Imagine the point $P$ sliding along the graph of the function $f(x)=\frac{(x-1)^2}{4}+3$ so that the $x$-coordinate of $P$ increases at a constant rate of 1.3 units per second.




a. What is the rate at which the length of $l$, the distance from point $P$ to the origin, is changing when the $x$-coordinate of point $P$ is 2?

Let $x_p$ represent the $x$-coordinate of the point $P$. The distance $l$ from point $P$ to the origin is given by

$$l=\sqrt{{x_p}^2+\left(\frac{(x_p-1)^2}{4}+3\right)^2}\,.$$

Differentiating $l$ with respect to $t$ gives

$$\frac{dl}{dt}=\frac{2x_p+\left(\frac{(x_p-1)^2}{4}+3\right)(x_p-1)}{2\sqrt{{x_p}^2+\left(\frac{(x_p-1)^2}{4}+3\right)^2}}\cdot\frac{dx_p}{dt}\,.$$

Evaluating ${}^{dl}/_{dt}$ for $x_p = 2$ and ${}^{dx_p}/_{dt} = 1.3$ gives

$$\frac{dl}{dt}=\frac{2(2)+\left(\frac{(2-1)^2}{4}+3\right)(2-1)}{2\sqrt{2^2+\left(\frac{(2-1)^2}{4}+3\right)^2}}\cdot(1.3)\approx 1.235\,.$$

Therefore, the length of $l$ is changing at a rate of 1.235 units per second when the $x$-coordinate of $P$ is 2.

b. What is the rate at which the angle $\theta$ in the figure above is changing when the $x$-coordinate of the point $P$ is 2?

We begin by computing the values of $l$ and $\theta$ when $x_p = 2$ and ${}^{dx_p}/_{dt} = 1.235$.

$$l=\sqrt{2^2+\left(\frac{(2-1)^2}{4}+3\right)^2}=3.816$$

$$\theta=\frac{\pi}{2}-\sin^{-1}\left(\frac{x_p}{l}\right)=\frac{\pi}{2}-\sin^{-1}\left(\frac{2}{3.816}\right)=1.019$$

We now define an equation that expresses the relationship between $\theta$, $x_p$, and $l$:

$$\sin\left(\frac{\pi}{2}-\theta\right)=\frac{x_p}{l}$$

$$\frac{d}{dt}\sin\left(\frac{\pi}{2}-\theta\right)=\frac{d}{dt}\left(\frac{x_p}{l}\right)$$

$$-\cos\left(\frac{\pi}{2}-\theta\right)\cdot\frac{d\theta}{dt}=\frac{\frac{dx_p}{dt}\cdot l-\frac{dl}{dt}\cdot x_p}{l^2}$$

$$\frac{d\theta}{dt}=\frac{\frac{dl}{dt}\cdot x_p-\frac{dx_p}{dt}\cdot l}{l^2\cos\left(\frac{\pi}{2}-\theta\right)}=\frac{1.235(2)-1.3(3.816)}{3.816^2\cos\left(\frac{\pi}{2}-1.019\right)}=-0.201$$

Therefore, the angle $\theta$ is decreasing at a rate of 0.201 radians per second when the $x$-coordinate of point $P$ is 2.

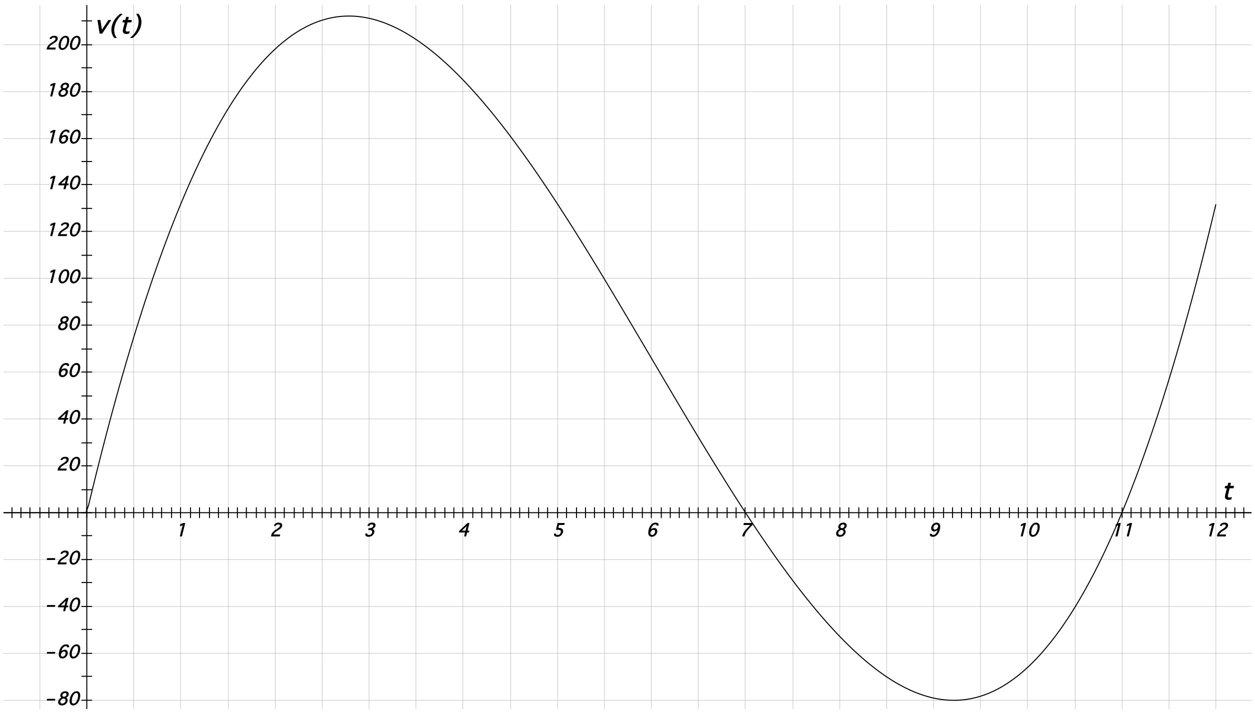
The objective of this investigation is to intuitively generate accumulation functions graphically and then capture this intuitive process into appropriate mathematical notation. At the conclusion of this investigation, students will have constructed a symbolic function rule that represents the accumulation of some quantity that is expressed in terms of how that quantity varies relative to some independent quantity.

This lesson begins by providing a conceptual foundation for the notion of Riemann sums that is *not* about area. The reality that the definite integral computes the signed area bound by the graph of a function and the $x$-axis over a specific interval of the function's domain is a byproduct of the fact that the definite integral quantifies the accumulation of some output quantity over some interval of the input domain. Therefore, we do not necessitate integration by discussing this geometric consequence of quantifying accumulation. This investigation lays the conceptual foundation for the Fundamental Theorem of Calculus by maintaining focus on the relationship between accumulation and rate of change.

We introduce the concept of an accumulation function (and ultimately the indefinite integral) by prompting students to approximate the value of some quantity $f(x)$ if all that is known is an initial value and the rate at which $f(x)$ changes as $x$ varies (i.e., the derivative of $f$). Students initially take a purely graphical approach to constructing a function that approximates the antiderivative of a function $f'$. Students then formalize their graphical method into a function rule that gives the accumulation of $f'$ at any specific value of the input quantity. As we mentioned above, this instructional progression seeks to ultimately provide a foundation for students to achieve a mature understanding of the Fundamental Theorem of Calculus.

The understandings promoted in this investigation were heavily informed by the research of Patrick Thompson and Jason Silverman (e.g., Thompson, 1994; Thompson & Silverman, 2008).

1. Chloe decides to go for a run before school. She starts her run from home. The graph of the function $v$ below represents the relationship between Chloe's velocity (in meters per minute) as she runs and the number of minutes $t$ elapsed since she started running.




a. Pick a point on the graph above and explain the meaning of its coordinates.
The point (5.50, 100.05) is on the graph of $v$. The coordinates of this point mean that Chloe's velocity was 100.05 meters per minute 5.50 minutes after she started running. Since Chloe does not actually have a velocity at this *specific* moment in time (because a velocity is a multiplicative comparison of a *change* in distance and a corresponding *change* in time), it is more accurate to say that Chloe is essentially running at a constant speed of 100.05 meters per minute over a *very* small interval of time around $t = 0.50$.
It is essential that you do not accept as a response to this question the claim that the $y$-coordinate represents Chloe's velocity *at a specific moment in time*. Previous investigations emphasized the notion that "instantaneous rate of change" is an abstract idea. Remind students that any rate of change is a multiplicative comparison of the change in the measure of an input quantity and the change in the measure of an output quantity. Students should therefore recognize that the $y$-coordinate of the point they select represents the constant velocity that Chloe runs over a *very* small interval of time around the $x$-coordinate. Interpreting the output values of a derivative function in this way lays the foundation for Riemann sums, developed later in this investigation.

b. Explain how you might approximate Chloe's distance (in meters) from home after she has been running for one minute?
Students will likely struggle with this task. Ask them to articulate what is difficult about approximating Chloe's distance from home after she has been running for one minute. Students should recognize that approximating the value of this quantity is challenging because Chloe's velocity is always changing. If her velocity was constant, one could compute the distance she travels by multiplying this constant velocity by 1, the change in the number of minutes she has been running. This recognition provides an incentive for approximating the behavior of the function $v$ over the interval [0, 1] with a constant function over this interval. Students will have several approaches to determining a constant velocity to approximate Chloe's changing velocity over the interval [0, 1]. Do not get too bogged down discussing these approaches. What matters

here is that students understand that they need to approximate the changing value of $v(t)$ as $t$ varies from 0 to 1 with a *constant value* over this interval.

c. Using the graph of $v$, approximate Chloe's distance (in meters) from home for the following number of minutes elapsed since she started running:
Students should employ the method they described in Part (a). While responding to Part (i), some students will be tempted to approximate Chloe's velocity over the interval [0, 3] with a single constant velocity. If students decide to do this, ask them how they might improve the value of their approximation. Support students in recognizing that they can approximate Chloe's changing velocity with a constant velocity over each one-minute interval and then sum the products of these constant velocities and the change in time over which they each occur. Asking students to represent their approximations using mathematical notation (function and summation notation in particular) will require them to be systematic in how they determine the constant velocity that approximates Chloe's changing velocity over each one-minute interval.

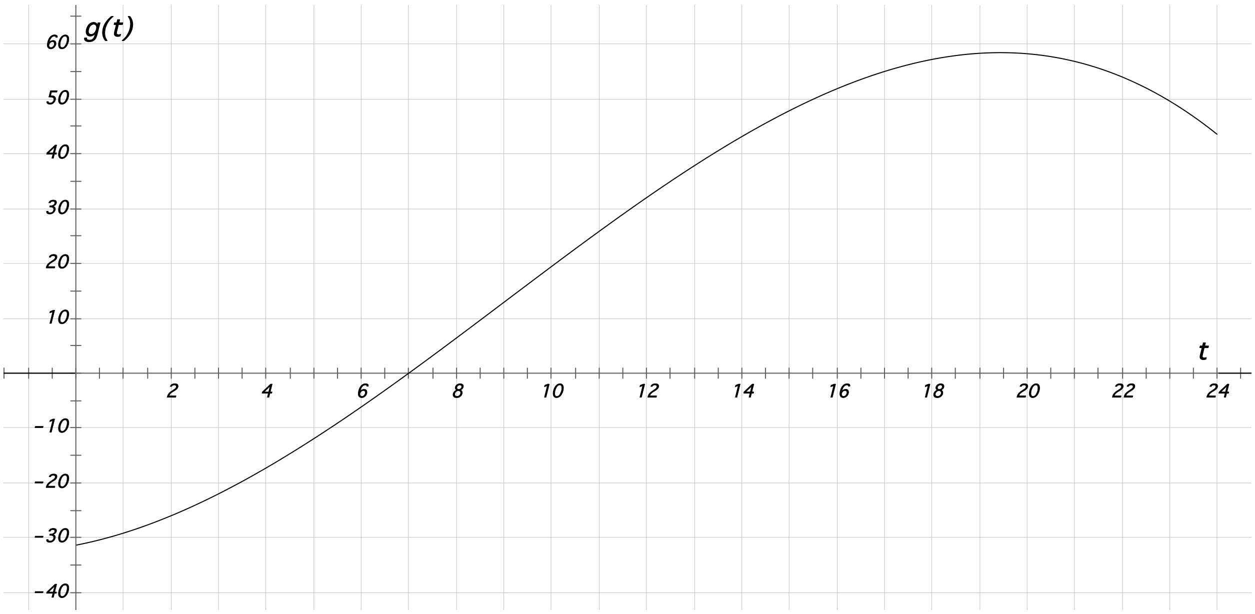
i. 3 minutes ($t = 3$).
Three minutes after Chloe started running, she was approximately $v(0)\cdot 1 + v(1)\cdot 1 + v(2)\cdot 1 \approx 330$ meters from home.

ii. 7 minutes ($t = 7$).
Seven minutes after Chloe started running, she was approximately $v(0)\cdot 1 + v(1)\cdot 1 + v(2)\cdot 1 + v(3)\cdot 1 + v(4)\cdot 1 + v(5)\cdot 1 + v(6)\cdot 1 \approx 924$ meters from home.

iii. 12 minutes ($t = 12$).
Three minutes after Chloe started running, she was approximately $\sum_{i=1}^{12} v(i-1)\cdot 1 \approx 726$ meters from home. One can also express the approximation as $\sum_{i=0}^{11} v(i)\cdot 1 \approx 726$ meters. (Note that these values reflect using a left-had Riemann sum.) Encourage students to express their approximation in summation notation, not just a numerical value.

d. Summarize the method you used in Part (c) to approximate Chloe's distance (in meters) from home 3 minutes, 7 minutes, and 12 minutes since she started running. How might you improve the accuracy of your approximations?
Approximating Chloe's distance (in meters) from home at the moments in time specified in Part (c) involved: (1) approximating Chloe's changing velocity over each successive one-minute interval with a constant velocity; (2) determining the change in Chloe's distance from home over each one-minute interval by multiplying the constant velocity that approximates $v(t)$ over each interval by 1, the change in the number of minutes since Chloe started running; and (3) summing the changes in Chloe's distance from home over each one-minute interval. The value of the approximations can be improved by decreasing the size of the interval of time over which Chloe's changing velocity is approximated with a constant velocity.

2. Everett decides to invest \$1,600 in the stock market. The graph of the function $g$ below represents the relationship between the rate of change in the value of stocks Everett purchased (in dollars per month) and the number of months $t$ elapsed since Everett purchased the stocks.




a. Pick a point on the graph above and explain the meaning of its coordinates.
The point (12, 32) appears to be on the graph of $g$. The coordinates of this point mean that the value of Everett's stocks was changing at a rate of \$32 per month 12 months after he invested. Since there is no rate of change in the value of Everett's stocks with respect to the number of months elapsed since he purchased the stocks at any *specific* moment in time (because a rate of change is a multiplicative comparison of a *change* in the measure of the input quantity and a corresponding *change* in the measure of the output quantity), it is more accurate to say that the value of Everett's stocks changed at a constant rate of \$32 per month over a very small interval of time around 12 months after Everett invested in the stock market.

b. Explain how you might approximate the value of Everett's stocks two months after he purchased them?
Students will likely struggle with this task. Ask them to articulate what is difficult about approximating the value of Everett's stocks two months after he purchased them. Students should recognize that approximating the value of this quantity is challenging because rate of change of the value of Everett's stocks with respect to the number of months elapsed since he purchased them is always changing. If this rate of change was constant, one could compute the value of Everett's stocks by multiplying this constant rate of change by 2, the change in the number of months elapsed since Everett purchased the stocks. It is therefore reasonable to approximate the behavior of the function $g$ over the interval [0, 2] with a constant function over this interval. It is important for students to understand that they need to approximate the changing value of $g(t)$ as $t$ varies from 0 to 2 with a *constant value* over this interval.

c. Using the graph of $g$, approximate the value of Everett's stocks for the following number of months elapsed since he invested in the stock market:
Students should employ the method they described in Part (a). While responding to Part (i), some students will be tempted to approximate the rate of change of the value of Everett's stocks over the interval [0, 6] with a single constant rate of change. If students decide to do this, ask them how they might improve the value of their approximation. Support students in recognizing that they can approximate the changing rate of change of the value of Everett's stocks with a constant rate of change over each two-month interval and then sum the products of these constant rates of change and the change in time over which they each occur. Asking students to represent their

approximations using mathematical notation (function and summation notation in particular) will require them to be systematic in how they determine the constant rate of change that approximates the changing rate of change of the value of Everett's stocks over each two-month interval.

i. 6 months ($t = 6$).
Six months after Everett invested in the stock market, the value of the stocks he purchased (in dollars) was approximately $1600 + g(0)\cdot 2 + g(2)\cdot 2 + g(4)\cdot 2 \approx \$1450$.

ii. 14 months ($t = 14$).
Fourteen months after Everett invested in the stock market, the value of the stocks he purchased (in dollars) was approximately $1600+\sum_{i=1}^{7} g\left(2(i-1)\right)\cdot 2 \approx 1550$ or

$1600+\sum_{i=0}^{6} g(2i)\cdot 2 \approx 1550$. (Note that these values reflect using a left-had Riemann sum.)

Encourage students to express their approximation in summation notation, not just a numerical value.

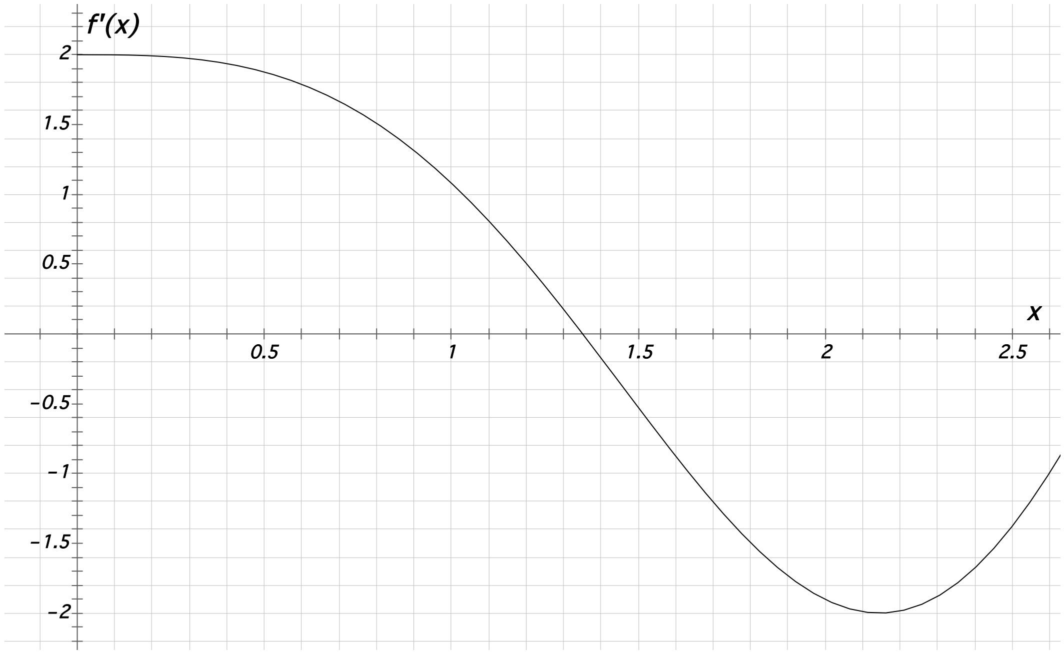
iii. 20 months ($t = 20$).
Twenty months after Everett invested in the stock market, the value of the stocks he purchased (in dollars) was approximately $1600+\sum_{i=1}^{10} g\left(2(i-1)\right)\cdot 2 \approx 1860$ or

$1600+\sum_{i=0}^{9} g(2i)\cdot 2 \approx 1860$. (Note that these values reflect using a left-had Riemann sum.)

Encourage students to express their approximation in summation notation, not just a numerical value.

d. Summarize the method you used in Part (c) to approximate the value of Everett's stocks 6 months, 14 months, and 20 months since he invested in the stock market. How might you improve the accuracy of your approximations?
Approximating the value of Everett's stocks (in dollars) at the moments in time specified in Part (c) involved: (1) approximating the changing rate of change of the value of Everett's stocks over each successive two-month interval with a constant rate of change; (2) determining the change in the value of Everett's stocks over each two-month interval by multiplying the constant rate of change that approximates $g(t)$ over each interval by 2, the change in the number of months elapsed since Everett purchased the stocks; and (3) summing the changes in the value of Everett's stocks over each two-month interval. The value of the approximations can be improved by decreasing the size of the interval of time over which we approximate the changing rate of change of the value of Everett's stocks with a constant rate of change.

3. The following is a graph of the function $f'$.




a. The point (1.10, 0.80) is on the graph of $f'$. Explain what these coordinates convey about the function $f$.
The coordinates (1.10, 0.80) on the graph of $f'$ mean that the local constant rate of change of $f(x)$ with respect to $x$ around $x = 1.10$ is 0.80.

b. Provided $f(0) = 0.2$, use the graph of $f'$ to approximate the value of $f(0.5)$. Explain how you determined your approximation.
An approximation of $f(0.5)$ is given by $0.5 \cdot f'(0) = 1$. We use the value $f'(0)$ as the constant rate of change that approximates the changing rate of change of $f(x)$ with respect to $x$ over the interval $0 \le x \le 0.5$. We then multiply this constant rate of change by 0.5, the interval over which we assume a constant rate of change of $f'(0)$. This product represents an approximation of the change in the value of $f(x)$ on the interval [0, 0.5].

c. Provided $f(0) = 0.2$, use the graph of $f'$ to approximate the following values. Express your approximations in numerical form and using summation notation.
The approximate values below are based on computing a left-hand Riemann sum. Students do not need to use a left-hand sum; their ability to answer this question does not rely upon having been exposed to the idea of Riemann sums. Accept more "approximate" values than the ones given below, which were computed with knowledge of the function definition of $f'$. It is important that you direct students' attention to the *process* of generating these approximations; the value of the approximations themselves is rather inconsequential.

i. $f(1)$
$f(1) \approx f(0) + 0.5 \cdot f'(0) + 0.5 \cdot f'(0.5) \approx 2.14.$

$$f(1) \approx f(0) + \sum_{i=1}^{2} f\left((i-1)0.5\right) \cdot 0.5 \approx 2.14.$$

ii. $f(1.5)$
$f(1.5) \approx f(0) + 0.5 \cdot f'(0) + 0.5 \cdot f'(0.5) + 0.5 \cdot f'(1) \approx 2.68.$

$$f(1.5) \approx f(0) + \sum_{i=1}^{3} f\left((i-1)0.5\right) \cdot 0.5 \approx 2.68.$$

iii. $f(2)$

$f(2) \approx f(0) + 0.5 \cdot f'(0) + 0.5 \cdot f'(0.5) + 0.5 \cdot f'(1) + 0.5 \cdot f'(1.5) \approx 2.42.$

$$f(2) \approx f(0) + \sum_{i=1}^{4} f\left((i-1)0.5\right) \cdot 0.5 \approx 2.42.$$

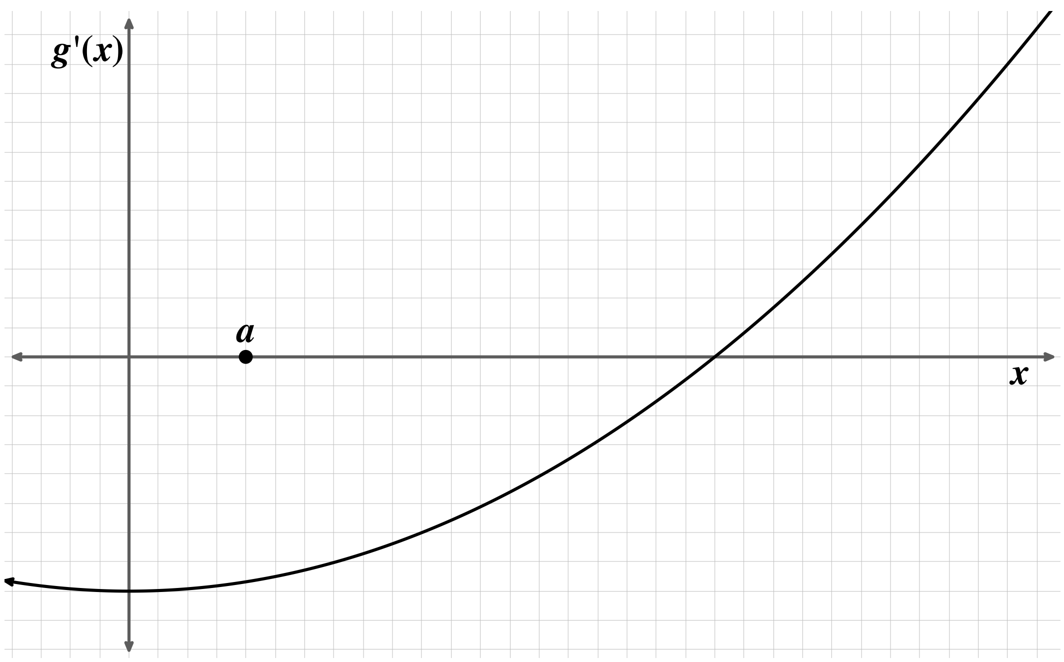
iv. $f(2.5)$

$f(2.5) \approx f(0) + 0.5 \cdot f'(0) + 0.5 \cdot f'(0.5) + 0.5 \cdot f'(1) + 0.5 \cdot f'(1.5) + 0.5 \cdot f'(2) \approx 1.46.$

$$f(2.5) \approx f(0) + \sum_{i=1}^{5} f\left((i-1)0.5\right) \cdot 0.5 \approx 1.46.$$

d. Summarize the method you used to approximate the various values of $f(x)$ in Part (c). How might you improve the accuracy of your approximations?
Approximating the values of $f(x)$ in Part (c) involved: (1) approximating the changing rate of change of $f(x)$ with respect to $x$ over successive intervals of length 0.5 with a constant rate of change; (2) determining the change in the value of $f(x)$ over each of these intervals by multiplying the constant rate of change that approximates $f(x)$ over each interval by 0.5, the length of each interval; and (3) summing the changes in the value of $f(x)$ over each interval of length 0.5. The value of the approximations can be improved by decreasing the size of the interval of $x$ over which we approximate the changing rate of change of $f(x)$ with a constant rate of change.

4. The following is a graph of the function $g'$.




a. The input value $x = a$ is in the domain of the function $g'$. Explain the meaning of $g'(a)$.
The value $g'(a)$ represents the local constant rate of change of $g(x)$ with respect to $x$ on a very small interval around $x = a$. In other words, $g'(a)$ represents the limiting value of the average rate of change of $g(x)$ with respect to $x$ over the interval $[a, a + \Delta x]$ as $\Delta x$ approaches zero.

b. Write an expression that represents the approximate value of $g(a + \Delta x)$ for some fixed $\Delta x$. Feel free to draw on the graph of $g'$ to organize/represent your thinking.
Students' solutions may vary. If one uses the value $g'(a)$ as an approximation for the changing rate of change of $g(x)$ with respect to $x$ over the interval $[a, a + \Delta x]$, then the solution is $g(a + \Delta x) \approx g(a) + g'(a) \cdot \Delta x$.

c. Write an expression that represents the approximate value of $g(a + n\cdot\Delta x)$ for some fixed $\Delta x$ and some whole number $n$. Feel free to draw on the graph of $g'$ to organize/represent your thinking.
Students' solutions may vary. If one uses the value $g'(a + (i - 1)\Delta x)$ as an approximation for the changing rate of change of $g(x)$ with respect to $x$ over the interval $[a + (i - 1)\Delta x, a + i\Delta x]$, for $i \in \{1, 2, \ldots, n\}$, then the solution is $g(a+\Delta x) \approx g(a) + \sum_{i=1}^{n} g'\big(a+(i-1)\Delta x\big)\cdot\Delta x$.

d. Write an expression that represents the approximate value of $g(x)$ for a generic value $x$ in the domain of $g$.
The expression $\left\lfloor \frac{x-a}{\Delta x} \right\rfloor$ represents the complete number of $\Delta x$-intervals between $x$ and $a$. The expression $g(a) + \sum_{i=0}^{\left\lfloor \frac{x-a}{\Delta x} \right\rfloor} g'\big(a+(i-1)\Delta x\big)\cdot\Delta x$ represents an approximate value of $g\left(a+\left\lfloor \frac{x-a}{\Delta x} \right\rfloor\Delta x\right)$.
The approximate change in the value of $g(x)$ from the input value $a+\left\lfloor \frac{x-a}{\Delta x} \right\rfloor\Delta x$ to the current value of $x$ is given by $g'\left(a+\left\lfloor \frac{x-a}{\Delta x} \right\rfloor\Delta x\right)\left(x-\left\lfloor \frac{x-a}{\Delta x} \right\rfloor\Delta x\right)$. Therefore, the approximate value of $g(x)$ is given by $g(x) \approx g(a) + \sum_{i=0}^{\left\lfloor \frac{x-a}{\Delta x} \right\rfloor} g'\big(a+(i-1)\Delta x\big)\cdot\Delta x + g'\left(a+\left\lfloor \frac{x-a}{\Delta x} \right\rfloor\Delta x\right)\left(x-\left\lfloor \frac{x-a}{\Delta x} \right\rfloor\Delta x\right)$.

e. How might you improve the accuracy of the approximation of the value of $g(x)$ you expressed in Part (d)? Write an expression that represents the *exact* value of $g(x)$.
One can improve the accuracy of the approximation of $g(x)$ from Part (d) by decreasing the magnitude of $\Delta x$. The following expression represents the exact value of $g(x)$:

$$g(x) = g(a) + \lim_{\Delta x\to 0} \sum_{i=0}^{\left\lfloor \frac{x-a}{\Delta x} \right\rfloor} g'\big(a+(i-1)\Delta x\big)\cdot\Delta x + g'\left(a+\left\lfloor \frac{x-a}{\Delta x} \right\rfloor\Delta x\right)\left(x-\left\lfloor \frac{x-a}{\Delta x} \right\rfloor\Delta x\right).$$

Since the expression $g'\left(a+\left\lfloor \frac{x-a}{\Delta x} \right\rfloor\Delta x\right)\left(x-\left\lfloor \frac{x-a}{\Delta x} \right\rfloor\Delta x\right)$ tends to zero as $\Delta x$ approaches zero, we can represent the exact value of $g(x)$ as

$$g(x) = g(a) + \lim_{\Delta x\to 0} \sum_{i=0}^{\left\lfloor \frac{x-a}{\Delta x} \right\rfloor} g'\big(a+(i-1)\Delta x\big)\cdot\Delta x.$$

We call the expression you wrote in Part (d) an ***accumulation function***. Provided information about the rate of change of Quantity A with respect to Quantity B (i.e., provided a derivative function), the outputs of the accumulation function represent the approximate measure of Quantity A associated with a particular measure of Quantity B. The following is a more precise definition of an accumulation function:

***Definition (Accumulation Function).*** Let $a$ be in the domain of a derivative function $f'$ and let $P = \{a, a + \Delta x, a + 2\Delta x, \ldots, a + n\Delta x, \ldots\}$ be a partition of $[a, \infty)$ for a constant value of $\Delta x$. Then the function that represents the approximate value of $f(x)$ as $x$ varies from $a$ to $\infty$ is given by

$$f(x) \approx f(a) + \sum_{i=0}^{\left\lfloor \frac{x-a}{\Delta x} \right\rfloor} f'\big(a+(i-1)\Delta x\big)\cdot\Delta x + f'\left(a+\left\lfloor \frac{x-a}{\Delta x} \right\rfloor\Delta x\right)\left(x-\left\lfloor \frac{x-a}{\Delta x} \right\rfloor\Delta x\right).$$

We call the expression you wrote in Part (e) an ***exact accumulation function***. We introduce a new notation to represent exact accumulation functions.

***Definition (Exact Accumulation Function).*** Let $a$ be in the domain of a derivative function $f'$ and let $P = \{a, a + \Delta x, a + 2\Delta x, \ldots, a + n\Delta x, \ldots\}$ be a partition of $[a, \infty)$ for some $\Delta x$. Then the function that represents the exact value of $f(x)$ as $x$ varies from $a$ to $\infty$ is given by

$$f(x) = f(a) + \lim_{\Delta x \to 0} \sum_{i=0}^{\left\lfloor \frac{x-a}{\Delta x} \right\rfloor} f'\left(a + (i-1)\Delta x\right) \cdot \Delta x \,.$$

A few remarks about notation are in order. Gottfried Leibniz developed the following notation to denote the derivative of $y$ with respect to $x$.

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}.$$

The Greek letter $\Delta$ (for difference) was changed to $d$ (for differential) to denote the value of $\Delta x$ was so small that $\Delta x$ and $\Delta y$ are essentially proportionally related. The notation for integration follows a similar theme. As $\Delta x$ approaches zero we see that the expression $f'\left(a + \left\lfloor \frac{x-a}{\Delta x} \right\rfloor \Delta x\right)\left(x - \left\lfloor \frac{x-a}{\Delta x} \right\rfloor \Delta x\right)$ in the definition of the accumulation function becomes infinitesimally small. This is why the definition of the exact accumulation function does not include it.

The summation in the expression that represents the exact accumulation function becomes an elongated "S" and the multiple of $\Delta x$ at the end becomes a $dx$ after the limit as $\Delta x$ approaches zero is taken. We call this elongated "S" an *integral.* Hence, we write

$$f(x) = \int_a^x f'(t)\,dt = f(a) + \lim_{\Delta x \to 0} \sum_{i=0}^{\left\lfloor \frac{x-a}{\Delta x} \right\rfloor} f'\left(a + (i-1)\Delta x\right) \cdot \Delta x$$

to represent the exact accumulation function of $f'$ on the domain $[a, \infty)$ assuming that $f'$ is defined for this subset of the real numbers. This expression should be thought of as the accumulation of $f'(t)$ as $t$ ranges from $a$ to $x$. The reason for the $t$ in $f'(t)$ is that the accumulation function is a function of $x$; the function $f'$ must preexist the variation of $x$ that produces the accumulation function. Hence, $t$ represents the independent variable for $f'$ that varies independently from $x$.